RIGIDITY AND REGULARITY OF CO-DIMENSION ONE SOBOLEV ISOMETRIC IMMERSIONS

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Abstract. We prove the developability and $C^{1,1/2}_{\rm loc}$ regularity of $W^{2,2}$ isometric immersions of n-dimensional domains into $\mathbb{R}^{n+1}$. As a conclusion we show that any such Sobolev isometry can be approximated by smooth isometries in the $W^{2,2}$ strong norm, provided the domain is C^1 and convex. Both results fail to be true if the Sobolev regularity is weaker than $W^{2,2}$.

1. Introduction

It has been known since at least the 19th century that any smooth surface with zero Gaussian curvature is locally ruled, i.e. passing through any point of the surface is a straight segment lying on the surface. Such surfaces were called developable surfaces. This terminology was used as an indication that any such surface is in isometric equivalence with the plane, i.e. any piece of it can be *developed* on the flat plane without any stretching or compressing. Meanwhile, it was already suspected that there exist somewhat regular surfaces applicable to the plane, but yet not developable (See [4] for a review of this question). Nevertheless, it was not until the work of John Nash at the zenith of the last century that the existence of such unintuitive phenomena was rigorously established.

In his pioneering work, Nash settled several questions. He established that any Riemannian manifold can be isometrically embedded in an Euclidean space [32]. Moreover, if the dimension of the space is large enough, this embedding can be done in a manner so that the diameter of the image is as small as one wishes. As for the lower dimensional embeddings, Nash [33] and Kuiper [22], established the existence of a C^1 isometric embedding of any Riemannian manifold into another manifold of one higher dimension. Their method, which is now famously re-cast in the framework of convex integration [12], involved iterated perturbations of a given short mapping of the manifold towards realizing an isometry.

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A surprising corollary of these results is the existence of a C^1 flat torus in $\mathbb{R}^3$ [1]. Another one is that there are C^1 isometric embeddings of the two dimensional unit sphere into three dimensional space with arbitrarily small diameter. By contrast, it was established by Hartman and Nirenberg that any flat C^2 surface in $\mathbb{R}^3$ must be developable [13], while Hilbert had already shown that any C^2 isometric immersion of the sphere must be a rigid motion. This latter result is a special case of a similar statement for any closed convex surface in $\mathbb{R}^3$, see [38, Chapter 12]. On the other hand, the former result was generalized by Pogorelov's for C^1 isometries with total zero curvature in [36, Chapter II] and [37, Chapter IX].

A natural question arises in this context for the analyst: What about isometric immersions of intermediate regularity, say of Hölder or Sobolev type? Regarding Hölder regularity, rigidity of $C^{1,\alpha}$ isometries of 2 dimensional flat domains has been established for $\alpha \geq 2/3$ [2, 3], while their flexibility in the sense of Nash and Kuiper is known for $\alpha < 1/7$ [3, 7]. The critical value for α is conjectured to be $1/2$ in this case. As for the regularity of Sobolev isometries, following the results of Kirchheim in [20] on $W^{2,\infty}$ solutions to degenerate Monge-Ampère equations (see Proposition 1.3), the rigidity of $W^{2,2}$ isometries of a flat domain was established in [35]. More precisely, it was established that such mappings are developable in the classical sense, i.e.

Theorem 1.1 (Pakzad [35])**.** *Let $v \in W^{2,2}(\Sigma, \mathbb{R}^3)$ be an isometric immersion, where Σ is a bounded Lipschitz domain in $\mathbb{R}^2$. Then $v \in C^{1,1/2}_{\mathrm{loc}}(\Sigma, \mathbb{R}^2)$. Furthermore, for every point of x, either there exists a neighborhood of x, or a unique segment passing through x and joining $\partial\Sigma$ at both ends, on which ∇v is constant.*

Remark 1.2. It can be shown that this statement is actually valid for all bounded open sets $\Sigma \subset \mathbb{R}^2$, i.e. without any assumption on the regularity of the boundary. All one must prove is that the constancy segments, whose existence are locally established, can be extended all the way to the boundary one step at a time. Assuming the existence of any supposedly maximal constancy segment which does not reach the boundary, a contradiction could be achieved by creating a Lipschitz domain $\Sigma' \subset \Sigma$ including the closure of that segment and applying Theorem 1.1 to Σ'. In the same manner, the regularity assumption on $\partial\Omega$ in Theorem 1.4 can be removed.

To put this result in context, it is noteworthy that a $W^{2,2}$ function on a two dimensional domain fails barely to be C^1, but there is information available about second weak derivatives, and e.g. the Gaussian curvature of the image of a $W^{2,2}$ isometric immersion of a flat domain is identically zero as an L^1 function. This indicates that these isometries are far from the highly oscillatory solutions of Nash and Kuiper and hence possibly should behave in a rigid manner. Note that only the C^1 regularity result was stated in [35] and was a major ingredient of the proof, but the higher Hölder regularity announced here is an immediate consequence of the developability. In [31] it

was established that the C^1 regularity can be extended up to the boundary if the domain is of class $C^{1,\alpha}$. This does not hold true anymore for merely C^1 regular domains. Finally, the following proposition is a key step in establishing the above rigidity result and will be instrumental in proving Theorem 1.4.

Proposition 1.3 (Kirchheim [20], Pakzad [35]). *Let Σ be as above and let $f \in W^{1,2}(\Sigma, \mathbb{R}^3)$ be a map with almost everywhere symmetric singular (i.e. of zero determinant) gradient. Then $f \in C^0(\Sigma)$ and for every point $x \in \Sigma$, there exists either a neighborhood U of x, or a segment passing through it and joining $\partial\Sigma$ at its both ends, on which f is constant.*

It was proved furthermore in [35] that any $W^{2,2}$ isometry on a convex 2d domain can be approximated in strong norm by smooth isometries. This is a nontrivial result, since the usual regularization techniques fail due to the non-linearity of the isometry constraint. The idea was to make use of the developability structure of these mappings and reduce the approximation problem to the one about mollifying the expressions $R^T R'$ for the Darboux moving frames $R(t)$ along the curves orthogonal to the rulings. The convexity assumption is a technical one, and as shown by Hornung [14, 16], can be replaced by e.g. piece-wise C^1 regularity of the boundary, see also [15].

Sobolev isometries also arise in the study of nonlinear elastic thin films. Kirchhoff's plate model put forward in the 19th century [21] consists in minimizing the L^2 norm of the second fundamental form of isometric immersions of a 2d domain into $\mathbb{R}^3$ under suitable forces or boundary conditions. In other words, using the modern terminology, the space of admissible maps for this model is that of $W^{2,2}$ isometric immersions. Using the developability results mentioned above for this class of mappings, Hornung has studied the regularity of the critical points of the Kirchhoff's functional in [17]. For other applications in nonlinear elasticity of both the developability and density results for Sobolev isometric immersions of flat domains see [6, 24].

In a broader context, it was shown by Friesecke, James and Müller [10] that Kirchhoff's model is rigorously justified as the Γ-limit of the 3d nonlinear elasticity variational model in a certain energy scaling regime. Their approach has since been generalized to other scaling regimes and various models for rods, plates and shells have been hence derived [11, 8, 9, 27, 28, 29, 18]. A common feature of these models is the role played by isometric immersions or infinitesimal isometries of the given elastic body. Furthermore, applying the same methodology to the prestrained elasticity, the research on Sobolev isometries or infinitesimal isometries of arbitrary Riemannian manifolds comes to the front-line of research in elasticity of rigid or prestrained structures [30, 23, 25, 26].

In this paper, using geometric methods and induction, we will generalize the results of [35] to higher dimensions. Our first main result is the developability of Sobolev co-dimension one isometries of flat domains in $\mathbb{R}^n$:

Theorem 1.4. *Let $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, be an isometric immersion, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then $u \in C^{1,1/2}_{\text{loc}}(\Omega, \mathbb{R}^{n+1})$. Moreover, for every $x \in \Omega$, either ∇u is constant in a neighborhood of x, or there exists a unique $(n-1)$-dimensional hyperplane $P \ni x$ of $\mathbb{R}^n$ such that ∇u is constant on the connected component of x in $P \cap \Omega$.*

The interesting feature of this new result is that the Sobolev regularity $W^{2,2}$ is much below the required $W^{2,n+\varepsilon}$ for obtaining C^1 regularity. Note that it is a well-established fact in differential geometry that higher dimensional manifolds are generally more rigid [38]. Also, a $W^{2,2}$ regularity for an isometry of co-dimension 1 implies that all sectional curvatures of the image vanish as L^1 functions, and removes the possibility of conic singularities. However, similar to the 2d case, these naive heuristic observations are not of much help for rigorously establishing the result. An extra difficulty which comes in the way of the proof in dimensions higher than 2 is that the argument used in Lemma 2.1 of [35] to show the continuity of the derivatives of the given Sobolev isometry is no more generalizable to our case. Indeed, in [35], a very important first step of the proof of developability is to show the C^1 regularity. Here, on the other hand, we first show the developability of the mapping without having the C^1 regularity at hand. Our proof is based an induction on the dimension of slices of the domain and careful and detailed geometric arguments. Having established developability, the C^1 regularity (and better) follows in a straightforward manner.

The problem of regularity and developability of Sobolev isometric immersions of co-dimension $k > 1$ is more involved and could not be tackled through the methods discussed in this paper. In a forthcoming paper by Jerrard and the second author [19], another approach, more analytical in nature, is adapted to study this problem. It is based on the fact that the Hessian-rank equation

$$\operatorname{rank}(\nabla^2 v) \leq k \quad \text{a.e.} in \Omega \tag{1.1}$$

is satisfied by the components $v = u^j$ of such isometry. Note that this equation becomes the degenerate Monge-Ampère equation when $k = n-1$. Similar as in [35], regularity and developability of the Sobolev solutions to (1.1) directly implies the same results for the corresponding isometries of the same regularity. However, one loses some natural advantages when working with (1.1) rather than with the isometries themselves as done in the present paper: the solution v is no more Lipschitz and being just a scalar function, one loses the extra information derived from the length preserving properties of isometries.

The second main result of this paper concerns approximation of $W^{2,2}$ isometries by smooth ones:

Theorem 1.5. *Assume $\Omega \subset \mathbb{R}^n$ is a C^1 bounded convex domain and that $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ is an isometric immersion. Then there is a sequence of isometric immersions $u_m \in C^\infty(\overline{\Omega}, \mathbb{R}^{n+1})$ converging to u in $W^{2,2}$ norm.*

The main idea of the proof, similar as in the 2d case, is to mollify the curves which pass orthogonally through the constancy hyperplanes of Theorem 1.4 both in the domain and on the image. This latter problem, framed within the general isometry mollification problem, is still nonlinear. However, identifying these curves with suitable orthonormal moving Darboux frames $R(t) \in SO(n)$ and $\tilde{R}(t) = [(\nabla u)R(t), \mathbf{n}(t)] \in SO(n+1)$, where $\mathbf{n}$ is the unit normal to the image of the isometry in $\mathbb{R}^{n+1}$, we could linearize the problem by considering the curvature matrices $R^T R'(t) \in so(n)$ and $\tilde{R}^T \tilde{R}'(t) \in so(n+1)$ and recover an approximating sequence of moving frames through their regularization. Many technical details must nevertheless be taken care of in this process; in particular one must make sure that the mollified curves can be used to define new smooth isometries. Also, the mapping as a whole cannot be described by one single couple of such curves and the domain must be partitioned into suitable subdomains.

Remark 1.6. Neither the C^1 regularity nor the convexity of the boundary seems to be absolutely necessary for the density result to hold true (see e.g. [16] for finer results in 2d), but omitting these assumptions goes beyond the scope of our paper.

Remark 1.7. Both of the results in Theorems 1.4 and 1.5 are sharp in the sense that they fail to be true if the isometric immersion is only of class $W^{2,p}$ for $p < 2$. An immediate counterexample is the following isometric immersion $u : B^2 \times (0,1)^{n-2} \to \mathbb{R}^{n+1}$, whose image can be visualized as a family of cones over a hyperplane of dimension $n-2$:

$$u(r\cos\theta, r\sin\theta, x_3, \cdots, x_n) := (\frac{r}{2}\cos(2\theta), \frac{r}{2}\sin(2\theta), \frac{\sqrt{3}}{2}r, x_3, \cdots, x_n).$$

The paper is organized as follows: In Section 2 we will review some basic analytic properties of isometric immersions with second order derivatives. Section 3 is dedicated to the proof of Theorem 1.4. In Section 4 we will show that smooth isometric immersions are strongly dense in the space of $W^{2,2}$ isometric immersions from a domain of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. The proof of Lemma 4.13, which is a crucial and difficult step in establishing the density result is for the convenience of the reader detailed in Appendix A.

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2. Preliminaries.

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n, n \geq 2$. We define the class of Sobolev isometric immersions from Ω to $\mathbb{R}^{n+1}$ as,

$$I^{2,2}(\Omega, \mathbb{R}^{n+1}) := \{u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) : (\nabla u)^T \nabla u = \mathrm{I} \text{ a.e.}\}. \tag{2.1}$$

Note that the condition $(\nabla u)^T \nabla u = \mathrm{I}$ implies that u is Lipschitz continuous, thus,

$$u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1}). \tag{2.2}$$

Given $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, let $u^j, 1 \leq j \leq n+1$, be the j-th component of u and let $u_{,i} = \partial u/\partial x_i, 1 \leq i \leq n$, be the partial derivative of u in the $\mathbf{e}_i$ direction. Throughout the paper we will use the same notation for all functions.

For a.e. $x \in \Omega$, consider the cross product

$$\mathbf{n}(x) = u_{,1}(x) \times \cdots \times u_{,n}(x).$$

That is, $\mathbf{n}(x)$ is the unique unit vector orthogonal to $u_{,i}(x)$ for all $1 \leq i \leq n$ such that

$$\mathbf{n}(x), u_{,1}(x), \cdots, u_{,n}(x)$$

form a positive basis of $\mathbb{R}^{n+1}$.

Note that $\mathbf{n}$ can also be identified as differential forms: consider the 1-form,

$$\omega_i = \sum_{j=1}^{n+1} u^j_{,i} dx_j.$$

Then

$$\mathbf{n} = *(\omega_1 \wedge \cdots \wedge \omega_n). \tag{2.3}$$

because for any $\xi \in \bigwedge^1(\mathbb{R}^{n+1})$,

$$\langle \xi, \mathbf{n} \rangle = \langle \xi, *(\omega_1 \wedge \cdots \wedge \omega_n) \rangle = \xi \wedge \omega_1 \wedge \cdots \wedge \omega_n = \det[\xi, u_{,1}, \cdots, u_{,n}].$$

Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1})$, it follows from (2.3) that $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{R}^{n+1})$.

As u is isometric immersion, $\langle u_{,i}, u_{,j}\rangle = \delta_{ij}$ for all $1 \le i, j \le n$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, we can differentiate using the product rule to obtain,

$$\langle u_{,ik}, u_{,j}\rangle + \langle u_{,i}, u_{,jk}\rangle = 0 \quad \text{a.e.} \tag{2.4}$$

Permutation of indices i, j, k yields,

$$\langle u_{,ij}, u_{,k}\rangle + \langle u_{,i}, u_{,kj}\rangle = 0 \quad \text{a.e.} \tag{2.5}$$

$$\langle u_{,ki}, u_{,j}\rangle + \langle u_{,k}, u_{,ji}\rangle = 0 \quad \text{a.e.} \tag{2.6}$$

Using the fact that $u_{,ij} = u_{,ji}$ for all i, j, we add (2.4) and (2.5), then subtract (2.6) to obtain,

$$\langle u_{,i}, u_{,jk}\rangle = 0 \quad \text{a.e.} \quad \text{for all} \quad 1 \le i, j, k \le n. \tag{2.7}$$

Since for a.e. points in the domain, $\mathbf{n}, u_{,1}, , , u_{,j}$ form a basis for $\mathbb{R}^{n+1}$, we can write,

$$u_{,jk} = \sum_{i=1}^{n} \langle u_{,jk}, u_{,i}\rangle u_{,i} + \langle u_{,jk}, \mathbf{n}\rangle \mathbf{n}.$$

(2.7) then gives,

$$u_{,jk} = \langle u_{,jk}, \mathbf{n}\rangle \mathbf{n} \quad \text{a.e.} \quad \text{for all} \quad 1 \le j, k \le n. \tag{2.8}$$

Note that $A_{jk} := \langle u_{,jk}, \mathbf{n}\rangle$ is the element in row j and column k of the second fundamental form A, which is an $n \times n$ matrix. In particular, (2.8) holds for each component of $u_{,jk}$ and $\mathbf{n}$, i.e.,

$$u^{\ell}_{,jk} = A_{jk}\mathbf{n}^{\ell} \quad \text{for all} \quad 1 \le \ell \le n+1, \quad 1 \le j, k \le n.$$

Thus, the Hessian of u^{ℓ} satisfies,

$$\nabla^2 u^{\ell} = \mathbf{n}^{\ell} A, \quad 1 \le \ell \le n+1. \tag{2.9}$$

Lemma 2.1. *The second fundamental form $A \in M^{n\times n}$ has the following properties,*

$$\frac{\partial A_{ij}}{\partial x_k} = \frac{\partial A_{ik}}{\partial x_j} \quad \textit{in distributional sense for all} \quad 1 \le i, j, k \le n, \tag{2.10}$$

and

$$A_{ij}A_{kl} - A_{il}A_{kj} = 0 \quad \textit{for all} \quad 1 \le i, j, k, l \le n. \tag{2.11}$$

Proof. For a smooth immersion $v : \Omega \to \mathbb{R}^{n+1}$, not necessarily isometric, let $g_{ij} = \langle v_{,i}, v_{,j}\rangle$ be the first fundamental forms, then by differentiating g_{ij} twice,

$$g_{ij,kl} = \langle v_{,ikl}, v_{,j}\rangle + \langle v_{,ik}, v_{,jl}\rangle + \langle v_{,il}, v_{,jk}\rangle + \langle v_{,i}, v_{,jkl}\rangle.$$

The summation over the proper permutations of i, j, k, l yields

$$g_{ij,kl} + g_{kl,ij} - g_{il,kj} - g_{kj,il} = -2\langle v_{,ij}, v_{,kl}\rangle + 2\langle v_{,il}, v_{,kj}\rangle. \tag{2.12}$$

Given any other smooth immersion $w : \Omega \to \mathbb{R}^{n+1}$, the following identity is also obvious,

$$\langle v_{,ij}, w\rangle_{,k} - \langle v_{,ik}, w\rangle_{,j} = \langle v_{,ij}, w_{,k}\rangle - \langle v_{,ik}, w_{,j}\rangle. \tag{2.13}$$

Now we let a sequence of smooth immersions $u_m \to u$ in $W^{2,2}(\Omega, \mathbb{R}^{n+1})$ and $\mathbf{n}_m \to \mathbf{n}$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$. Writing the left hand sides of (2.12) and (2.13) as distributional derivatives and passing to the limit we get,

$$0 = -2\langle u_{,ij}, u_{,kl}\rangle + 2\langle u_{,il}, u_{,kj}\rangle. \tag{2.14}$$

because $\langle u_{,i}, u_{,j}\rangle = \delta_{ij}$ for all i, j. In addition, since $\mathbf{n}$ is a unit vector, $\langle \mathbf{n}_{,k}, \mathbf{n}\rangle = 0$. Then by (2.8), $\langle u_{,ij}, \mathbf{n}_{,k}\rangle = 0$ for all i, j, k, thus,

$$\langle u_{,ij}, \mathbf{n}\rangle_{,k} - \langle u_{,ik}, \mathbf{n}\rangle_{,j} = 0 \tag{2.15}$$

The two identities in the lemma follow easily from $A_{ij} = \langle u_{,ij}, \mathbf{n}\rangle$, (2.14), and (2.15). The proof is complete. □

Corollary 2.2. *The second fundamental form A satisfies* $\operatorname{rank} \mathrm{A} \le 1$ *and A is symmetric a.e. in Ω. Moreover, the Hessian of each component of u satisfies* $\operatorname{rank} \nabla^2 u^\ell \le 1$ *for all $1 \le \ell \le n+1$ a.e. on Ω.*

Proof. By identity (2.11), all 2×2 minors of A vanish, hence the rank of A is less than or equal to 1. By (2.9), $\operatorname{rank} \nabla^2 u^\ell = n^\ell \operatorname{rank} A \le 1$ and A is symmetric a.e. since $\nabla^2 u^\ell$ is symmetric a.e. The proof is complete. □

3. Developability and Regularity

Our first main result- Theorem 1.4- follows from the following proposition:

Proposition 3.1. *Let $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let A be the second fundamental form of u. Let P_k be a k-dimensional plane of $\mathbb{R}^n$, $k \le n$. Suppose on $P_k \cap \Omega$ we have the following properties,*

(1) *There exists a sequence of smooth functions u^ϵ defined in the domain Ω such that*
$$\int_{P_k\cap\Omega} |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^k \to 0.$$
Here ∇u^ϵ, ∇u, $\nabla^2 u^\epsilon$ and $\nabla^2 u$ denote the full gradient with respect to the domain Ω.
(2) *The full gradient ∇u satisfies $\nabla u^T \nabla u = \mathrm{I}$ a.e. on $P_k \cap \Omega$.*
(3) $\nabla^2 u^\ell = \mathbf{n}^\ell A, 1 \le \ell \le n+1$ *a.e. on $P_k \cap \Omega$.*
(4) $\operatorname{rank} A \le 1$ *and A is symmetric a.e. on $P_k \cap \Omega$.*

Then $u \in C^{1,1/2}_{\rm loc}(P_k, \mathbb{R}^{n+1})$. Moreover, for every $x \in P_k \cap \Omega$, either ∇u is constant in a neighborhood in $P_k \cap \Omega$ of x, or there exists a unique $(k-1)$-dimensional hyperplane $P^x_{k-1} \ni x$ of P_k such that ∇u is constant on the connected component of x in $P^x_{k-1} \cap \Omega$.

The proof of this proposition is based on induction on lower dimensional slices. Before we prove Proposition 3.1, we will show that it implies Theorem 1.4.

Proof of Theorem 1.4. We simply take $k = n$ in Proposition 3.1, in which case $P_n \cap \Omega = \Omega$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, the convolution of u with the standard mollifier u^ϵ apparently satisfies assumption (1). By the fact that $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, $\nabla u^T \nabla u = \mathrm{I}$ a.e. in Ω, which is property (2). Property (3) follows from equation (2.9) and property (4) follows from Corollary 2.2 . Therefore, all the assumptions of Proposition 3.1 are satisfied, and hence the conclusion of Theorem 1.4 follows from the conclusion of Proposition 3.1. The proof is complete. □

Corollary 3.2. *Let $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, where Ω is a Lipschitz domain in $\mathbb{R}^n$. Then for every k dimensional slice $P_k \cap \Omega$, ∇u is constant either on k dimensional neighborhoods of $P_k \cap \Omega$, or constant on $k-1$-dimensional slice of $P_k \cap \Omega$.*

Proof. Since assumptions (1)-(4) of Proposition 3.1 are satisfied a.e. in Ω. By Fubini Theorem, assumptions (1)-(4) also holds in $\mathcal{H}^{n-k}$ a.e. k-dimensional slice. Thus the conclusion of Proposition 3.1 holds for a.e. k-dimensional slices. Since ∇u is continuous, by a simple approximation argument, it holds on *every* k-dimensional slices. The proof is complete. □

Assumptions (2) (3) and (4) regard the properties of isometric immersions, while (1) can be formulated for any general Sobolev function. This latter assumption is necessary for allowing the use of the chain rule which involves the full gradient even in lower dimensional slices. To be precise, we prove the following lemma which will play an important role everywhere in the proof of Proposition 3.1.

Lemma 3.3 (Chain Rule)**.** *Let $\Psi \in W^{1,2}(\Omega, \mathbb{R}^N)$, $N \geq 1$. Suppose for any k-dimensional domain $\Sigma \subset \Omega$ there exist a sequence of smooth functions $\Psi^\epsilon \in C^\infty(\Omega, \mathbb{R}^N)$ such that*

$$\int_\Sigma |\Psi^\epsilon - \Psi|^2 + |\nabla \Psi^\epsilon - \nabla \Psi|^2 \, d\mathcal{H}^k \to 0, \tag{3.1}$$

where $\nabla \Psi$ denotes the full gradient with respect to the domain Ω. Let $\mathbf{v}$ be any directional vector of Σ, then the chain rule,

$$\frac{d}{dt}\Big|_{t=0} \Psi(\cdot + t\mathbf{v}) = \nabla \Psi \mathbf{v}$$

holds in the weak sense over the domain Σ. In particular,

$$\Psi \in W^{1,2}(\Sigma, \mathbb{R}^N).$$

Proof. Let $\phi \in C_0^\infty(\Sigma)$, then,

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x+t\mathbf{v})\phi(x)d\mathcal{H}^k = -\int_\Sigma \Psi^\epsilon(x)\frac{d}{dt}\Big|_{t=0}\phi(x+t\mathbf{v})d\mathcal{H}^k.$$

Since Ψ^ϵ is smooth in Ω, we have,

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x+t\mathbf{v})\phi(x)d\mathcal{H}^k = \int_\Sigma \nabla\Psi^\epsilon(x)\mathbf{v}\phi(x)d\mathcal{H}^k.$$

By (3.1) we pass to the limit to conclude that,

$$\int_\Sigma \nabla\Psi(x)\mathbf{v}\phi(x)d\mathcal{H}^k = -\int_\Sigma \Psi(x)\frac{d}{dt}\Big|_{t=0}\phi(x+t\mathbf{v})d\mathcal{H}^k.$$

Thus the chain rule as stated in the Lemma hold in the weak sense over the domain Σ. The proof is complete. □

Remark 3.4. Note that the above lemma involves the *full* gradient of Ψ. The assumption $\Psi \in W^{1,2}(\Sigma,\mathbb{R}^k)$ by itself is not enough to conclude the chain rule.

3.1. **Base case- 2-dimensional slices.** Suppose for a 2-dimensional plane P_2 all the assumptions (1)-(4) in Proposition 3.1 are satisfied. Without loss of generality, we can assume P_2 is parallel to the space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Indeed, it is easy to see that assumption (1)-(4) in Proposition 3.1 are invariant under rotating the coordinate system. We denote P_2 by $P_{\mathbf{e}_1\mathbf{e}_2}$ to remind ourselves of this fact.

Let $f = \nabla u^\ell \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ for some arbitrary $1 \le \ell \le n+1$. Define,

$$g := (f^1, f^2)|_{P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega} \in W^{1,2}(P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega,\mathbb{R}^2).$$

Lemma 3.5. *Let C be a line segment in $P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega$ such that*

$$\int_C |f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2 d\mathcal{H}^1 \to 0, \tag{3.2}$$

$\operatorname{rank}\nabla f \le 1$ *and ∇f is symmetric for $\mathcal{H}^1$ a.e. points on C. Then if g is constant on C, so is f.*

Proof. Let $\mathbf{v}$ be the unit directional vector of C. Since $\mathbf{v}$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0, \cdots, 0).$$

Let $\tilde{\mathbf{v}} = (\mathbf{v}_1,\mathbf{v}_2)$, then the first two components of f satisfy $\nabla f^1 \cdot \mathbf{v} = \nabla g^1 \cdot \tilde{\mathbf{v}}$ a.e. on C and $\nabla f^2 \cdot \mathbf{v} = \nabla g^2 \cdot \tilde{\mathbf{v}}$ a.e. on C.

Since f satisfies the assumption of Lemma 3.3, the chain rule,

$$\frac{d}{dt}\Big|_{t=0} f(\cdot + t\mathbf{v}) = (\nabla f)\mathbf{v}$$

holds in the weak sense on C. In particular, it holds for it first two component f^1 and f^2 and, of course, g.

As g is constant on C,

$$0 = \frac{d}{dt}\Big|_{t=0} g(\cdot + t\tilde{\mathbf{v}}) \tag{3.3}$$

in the weak sense. Hence,

$$(\nabla g)\tilde{\mathbf{v}} = 0 \quad \text{a.e. on } C.$$

This implies,

$$\nabla f^1 \cdot \mathbf{v} = 0 \quad \text{and} \quad \nabla f^2 \cdot \mathbf{v} = 0 \quad \text{a.e. on } C.$$

For $z \in C$ such that $\nabla f^1(z) \cdot \mathbf{v} = 0$ and $\nabla f^2(z) \cdot \mathbf{v} = 0$, $\operatorname{rank} \nabla f(z) \leq 1$ and $\nabla f(z)$ is symmetric, we have two cases: 1)$\nabla f^1(z) \neq 0$ or $\nabla f^2(z) \neq 0$, 2)$\nabla f^1(z) = \nabla f^2(z) = 0$ In the first case, we can assume with loss of generality that $\nabla f^1(z) \neq 0$. Therefore, $\operatorname{rank} \nabla f(z) = 1$ and

$$\nabla f^i(z) = a^i_z \nabla f^1(z) \quad \text{for all } i > 1$$

It then follows that

$$\nabla f^i(z) \cdot \mathbf{v} = a^i_z \big(\nabla f^1(z) \cdot \mathbf{v}\big) = 0 \quad \text{for all } i > 1.$$

In the second case, by symmetry,

$$f^i_{,j}(z) = f^j_{,i}(z) = 0, \quad \text{for } j = 1, 2, \text{ and } i = 1, \cdots, n.$$

As $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0, \cdots, 0)$,

$$\nabla f^i(z) \cdot \mathbf{v} = 0 \quad \text{for all } i = 1, \cdots, n$$

Therefore, in either cases, we have proved

$$\nabla f^i \cdot \mathbf{v} = 0 \quad \text{a.e. on } C \quad \text{for all } i = 1, \cdots, n.$$

Therefore, f is constant on C by the chain rule in (3.3). The proof is complete. □

Corollary 3.6. *If g is constant on a 2-dimensional region U in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, f is constant on U as well.*

Proof. Observe that if U is a 2-dimensional region of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, which has strictly positive 2 Hausdorff measure, then the assumptions (1) and (4) of Proposition 3.1 imply,

$$\int_U |f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2 d\mathcal{H}^2 \to 0,$$

$\operatorname{rank} \nabla f \leq 1$ and ∇f is symmetric for $\mathcal{H}^2$ a.e. points on U. Thus the same argument for line segments in Lemma 3.5 gives for any directional vector $\mathbf{v}$ of U, $\nabla f^i \cdot \mathbf{v} = 0$ a.e. on U for all $i = 1, \cdots, n$, hence the chain rule implies f is constant on U. The proof is complete. □

Lemma 3.7. *Suppose assumptions (1)-(4) of Proposition 3.1 are satisfied on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Let $f = \nabla u^\ell \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^n)$ for some arbitrary $1 \le \ell \le n+1$. Then $f \in C^{0,1/2}_{\rm loc}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^n)$. Moreover, for every point of x, either there exists a neighborhood in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ of x, or a unique line segment in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends, on which f is constant.*

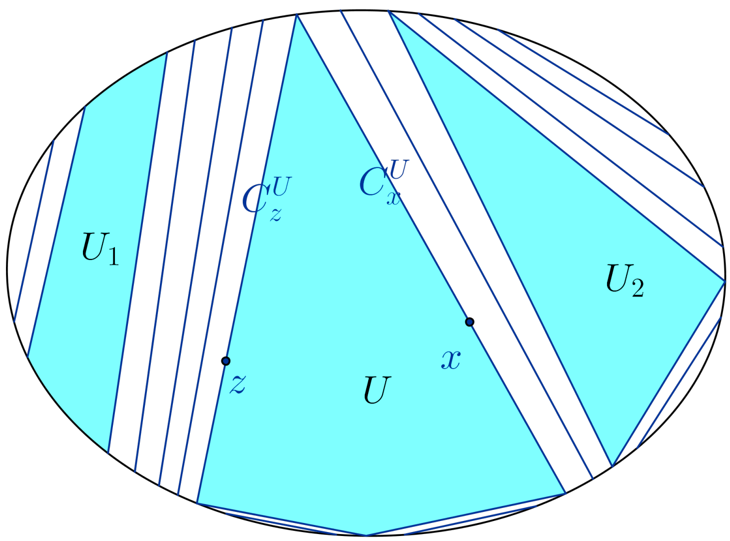

FIGURE 1. Inverse image of g in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$

Proof. By assumption (4) of Proposition 3.1, ∇f satisfies $\operatorname{rank} \nabla f \le 1$ and $\nabla f = \nabla^2 u^\ell$ is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Therefore, $g := (f^1, f^2)|_{P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega} \in W^{1,2}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^2)$ also satisfies $\operatorname{rank} \nabla g \le 1$ and ∇g is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. We employ [35], Proposition 1, which is cited above as Proposition 1.3. g satisfies the assumption of this proposition on the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ and hence the conclusions holds true for g. Suppose g is constant on some maximal connected neighborhood $U \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$; by continuity of g, it is also constant on its closure $\overline{U} \cap \Omega$. Now if $x \in \partial U \cap \Omega$, then x is not contained in a constant neighborhood of g, therefore by Proposition 1.3, there exists a unique line segment $C_x^U \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends on which g is constant, which implies $\partial U \cap \Omega \subset \bigcup_{x \in \partial U \cap \Omega} C_x^U$. Moreover, for $x, z \in \partial U \cap \Omega$, $C_x^U = C_z^U$ if $z \in C_x^U$ and $C_x^U \cap C_z^U \cap \Omega = \emptyset$ if $z \notin C_x^U$ (Figure 1). This follows from the fact that if g is constant on two such intersecting segments, it must be constant on their convex hull inside Ω too. On the other hand, suppose g is constant on some line segment C_x^U passing through $x \in \partial U \cap \Omega$ and joining $\partial\Omega$ at both end, since g is constant on $\overline{U}$ and C_x^U, which intersect at x, it must be constant on the convex hull of $\overline{U}$ and C_x^U inside Ω. But U is maximal, hence

$\bigcup_{x\in\partial U\cap\Omega} C_x^U \subset \partial U \cap \Omega$. Therefore,

$$\partial U \cap \Omega = \bigcup_{x\in\partial U\cap\Omega} C_x^U .$$

Let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ be such that g is not constant in a neighborhood of x_0. We can choose small enough $\delta > 0$ so that for any region U on which g is constant, the 2-dimensional ball $B^2(x_0,\delta) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ intersects ∂U at no more than *two* line segments belonging to ∂U. We use the fact that for any maximal constant region U, line segments in ∂U do not intersect inside Ω. If x_0 is at a positive distance of all constancy regions, the conclusion is trivial. The same is true if it lies on the boundary of one of a constancy region and yet is positively distant from all others. Suppose therefore that there is a sequence of maximal constant regions U_m converging to x_0 in distance, in which case there are two line segments $C_{x_1}^{U_m}$ and $C_{x_2}^{U_m}$ in ∂U_m whose angle (if they are nonparallel) or distance (if they are parallel) converges to zero, since both of these sequences of segments must converge to the same constancy segment passing through x_0. Then all the other line segments in ∂U_m must be arbitrarily close to $\partial\Omega$, we can again choose δ small enough so that $B^2(x_0,\delta)$ is away from $\partial\Omega$ and hence it does not intersect a third line segment in ∂U_m (Figure 2).

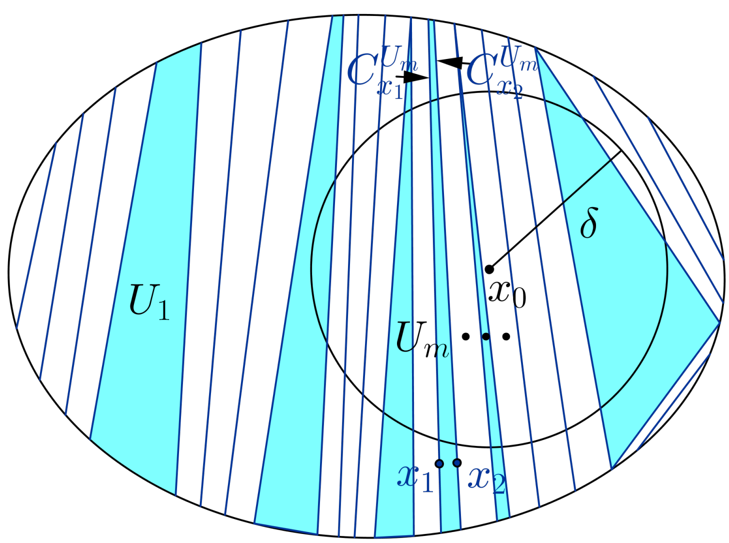

FIGURE 2. $B^2(x_0,\delta)$ intersects ∂U_m at two line segments

.

We now focus on $B^2(x_0,\delta) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. For any $x \in B^2(x_0,\delta)$, we want to construct a line segment C_x in $B^2(x_0,\delta)$ passing through x and joining $S^2(x_0,\delta)$ at both ends on which g is constant and $C_x \cap C_z \cap B^2(x_0,\delta) = \emptyset$ if $z \notin C_x$. For those x not contained in a constant region of g, this line segment is given automatically by Proposition 1.3. If x is contained in a constant

maximal region U of g, then it is constant on every line segment in U that passes through it so we have to choose the appropriate one: 1) If $B^2(x_0,\delta)$ intersect only one line segment C^U in Ω that belongs to ∂U, then we define C_x to be the line segments inside $B^2(x_0,\delta)$ passing through x and parallel to C^U; 2) If $B^2(x_0,\delta)$ intersects two line segments C_1^U, C_2^U in Ω that belongs to ∂U, let L_1 and L_2 be the two lines that contain C_1^U and C_2^U. If L_1 and L_2 are not parallel, let $O := L_1 \cap L_2$ and let C_x be the segment of the line passing through O and x inside $B^2(x_0,\delta)$. If L_1 and L_2 are parallel, then we let C_x be the line segment inside $B^2(x_0,\delta)$ passing through x and parallel to L_1. (Figure 3).

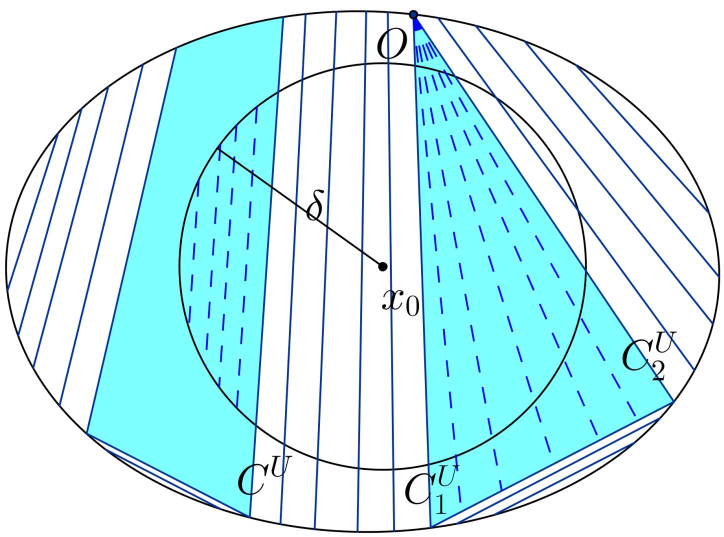

FIGURE 3. Construction of foliations.

In this way, we construct a family of line segments $\{C_x\}_{x\in B^2(x_0,\delta)}$ in $B^2(x_0,\delta)$ on which g is constant and $C_x \cap C_z \cap B^2(x_0,\delta) = \emptyset$ if $z \notin C_x$. For every $x \in B^2(x_0,\delta)$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector in $B^2(x_0,\delta)$ orthogonal to C_x. By making δ smaller we can make sure that none of the C_xs intersect inside $B^k(x_0,2\delta)$, and therefore we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector field inside the ball of radius δ. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

then has a unique solution $\gamma : (a,b) \to B^2(x_0,\delta)$ for some interval $(a,b) \subset \mathbb{R}$ containing 0. Moreover, if necessary by making δ smaller, $\cup\{C_{\gamma(t)}\}_{t\in(a,b)} = B^2(x_0,\delta)$. Therefore, $\{C_{\gamma(t)}\}_{t\in(a,b)}$ is a foliation of $B^2(x_0,\delta)$ (Figure 4).

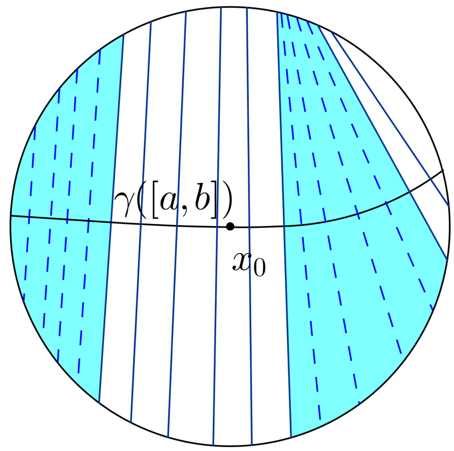

FIGURE 4. Foliations of $B^2(x_0,\delta)$.

We define the function $h : B^2(x_0,\delta) \to B^2(x_0,\delta)$ as

$$h(x) = \gamma(t) \quad \text{if} x \in C_{\gamma(t)}.$$

Since none of the $C_{\gamma(t)}$ intersect inside $B^2(x_0,\delta)$, h is well defined and h is constant along each $C_{\gamma(t)}$, i.e. $h^{-1}(\gamma(t)) = C_{\gamma(t)}$. Since γ is Lipschitz, h is Lipschitz as well. Moreover, since $|\gamma''(t)|$ is uniformly bounded, we have the Jacobian $J_h > C > 0$.

We now want to show the assumptions of Lemma 3.5 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. Let E_0 be the set of all $x \in B^2(x_0,\delta)$ such that rank $\nabla f(x) > 1$ or $\nabla f(x)$ is not symmetric. By assumption (4) of Proposition 3.1 on f, $|E_0| = 0$. As h is Lipschitz, we can apply the co-area formula to h to obtain,

$$\begin{aligned}
0 = \int_{E_0} J_h(x)dx \quad &= \int_\gamma \mathcal{H}^1\big(E_0 \cap h^{-1}(w)\big)d\mathcal{H}^1(w) \quad = \int_a^b \mathcal{H}^1\big(E_0 \cap h^{-1}(\gamma(t))\big)|\gamma'(t)|\,dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; = \int_a^b \mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big)|\gamma'(t)|\,dt.
\end{aligned}$$

Therefore, for a.e. $t \in (a,b)$, $\mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big) = 0$ since $|\gamma'| = 1$. Moreover, by change of variable formula,

$$\begin{aligned}
\int_{B^2(x_0,\delta)} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big)J_h \quad &= \int_\gamma \int_{h^{-1}(w)} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big)d\mathcal{H}^1 d\mathcal{H}^1(w) \\
&= \int_a^b \int_{h^{-1}(\gamma(t))} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big)d\mathcal{H}^1|\gamma'(t)|\,dt \\
&= \int_a^b \int_{C_{\gamma(t)}} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big)d\mathcal{H}^1|\gamma'(t)|\,dt.
\end{aligned}$$

Since J_h is bounded, together with assumption (1) in Proposition 3.1, we then have for a.e. $t \in (a,b)$,

$$\int_{C_{\gamma(t)}} \left(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\right) d\mathcal{H}^1 \to 0.$$

Therefore, the assumptions of Lemma 3.5 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. It follows that f is constant on $C_{\gamma(t)}$ for a.e. $t \in (a,b)$.

By choosing an initial value for γ arbitrary close to x_0 and applying the co-area formula in a similar manner we can make sure that f is of class $W^{1,2}$ on γ. Hence we conclude that f is $C^{0,1/2}$ on γ by the Sobolev embedding theorem. Let F be the set of $t \in (a,b)$ such that f is not constant along $C_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify f to be constant along $C_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^1(\bigcup\{C_{\gamma(t)} : t \in F\}) \leq 2\delta \sup J_h^{-1} \mathcal{H}^1(\{\gamma(t) : t \in F\}) = 2\delta \sup J_h^{-1} \mathcal{H}^1(F) = 0.$$

Hence f is $C^{0,1/2}$ up to modification of a set of measure zero in $B^2(x_0,\delta)$. Moreover, f is constant on $C_{\gamma(t)}$ for all t, which foliates $B^2(x_0,\delta)$. In addition, by Corollary 3.6, f is constant on every 2-dimensional region in $B^2(x_0,\delta)$ on which g is constant. Therefore, f is either constant on a line segment joining $\partial B^2(x_0,\delta)$ at both ends, or constant on a 2-dimensional region in $B^2(x_0,\delta)$. This proves Lemma 3.7 for the ball $B^2(x_0,\delta)$.

Now we prove the lemma for the entire domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Indeed, suppose for some $x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, x is not contained in a constant region of f. Then by what we have proved, f is constant on a line segment passing through x and joins the boundary of $B^2(x,\delta_x) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_x > 0$. Let $\overline{y_1y_2}$ be the largest line segment containing this segment on which f is constant. Suppose $y_1 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then from what we have proved, f is either constant on 2-dimensional regions or line segments passing through y_1 and joining the boundary of $B^2(y_1,\delta_{y_1}) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_{y_1} > 0$. First y_1 cannot be contained in a constant region of f, otherwise we can prolong the segment $[y_1,y_2]$. Thus, there must be a line segment $\overline{z_1z_2}$ passing through y_1 and joining the boundary of $B^2(y_1,\delta_{y_1})$ at both end on which f is constant. Second, $\overline{z_1z_2}$ cannot have the same direction as $\overline{y_1y_2}$, otherwise, we can again prolong the segment $\overline{y_1y_2}$. Then we consider the region Δ bounded by $\overline{y_2z_1}$, $\overline{z_1z_2}$ and $\overline{z_2y_2}$. Since g is constant on $\overline{y_1y_2}$ and $\overline{z_1z_2}$, by Theorem 1.1, g must be constant on Δ because no line segment can join the boundary of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through a point inside Δ without intersecting either $\overline{y_1y_2}$ or $\overline{z_1z_2}$ (Figure 5). Hence by Corollary 3.17, f is constant on Δ as well, contradiction to our assumption x is not contained in a constant region of f. The proof is complete. $\square$

Now we are ready to prove Proposition 3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Since we take $f = \nabla u^\ell$ for arbitrary $1 \leq \ell \leq n+1$, Lemma 3.7 gives all ∇u^ℓ are continuous on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ and constant either on 2-dimensional neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ joining $\partial\Omega$ at both ends. Therefore,

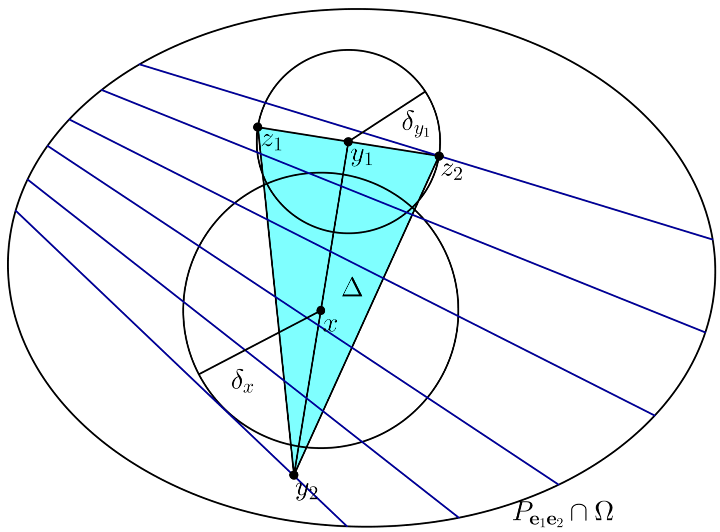

FIGURE 5.

what is left is to prove that they are constant on the *same* neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$.

Recall equation (2.3) that $\mathbf{n}$ is the wedge product of entries of ∇u, hence is continuous. Let

$$\Delta_\ell = \{x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega : \mathbf{n}^\ell(x) \neq 0\}.$$

Apparently each Δ_ℓ is open by continuity. Moreover, since $|\mathbf{n}| = 1$ everywhere,

$$\bigcup_{1 \le \ell \le n+1} \Delta_\ell = P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega.$$

Let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then $x_0 \in \Delta_\ell$ for some ℓ, without loss of generality, Δ_1. Then as the same argument as in the proof of Lemma 3.7, there exist $B^2(x_0, \delta) \subset \Delta_1$ for some $\delta > 0$, on which we can construct a foliation $\{C_{\gamma(t)}\}_{t \in (a,b)}$, i.e. $\cup \{C_{\gamma(t)}\}_{t \in (a,b)} = B^2(x_0, \delta)$ and $C_\gamma(t) \cap C_\gamma(t') \cap B^2(x_0, \delta) = \emptyset$ for $t' \neq t$. Moreover, $\nabla^2 u^1$ is constant on $C_\gamma(t)$ for every $t \in (a, b)$. Assumption (1) and (3) in Proposition 3.1, together with the same argument using co-area formula and change of variable formula as in the proof of Lemma 3.7 yield for a.e. $t \in (a, b)$

$$\int_{C_{\gamma(t)}} |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^1 \to 0$$

and $\nabla^2 u^\ell = (\mathbf{n}^\ell / \mathbf{n}^1) \nabla^2 u^1, 2 \le \ell \le n+1$ $\mathcal{H}^1$ a.e on $C_{\gamma(t)}$.

Let $\mathbf{v}$ be the directional vector of one such $C_{\gamma(t)}$, then the chain rule in Lemma 3.3 and the fact that ∇u^1 is constant on $C_{\gamma(t)}$ imply

$$0 = \frac{d}{dt}\Big|_{t=0} \nabla u^1(\cdot + t\mathbf{v}) = (\nabla^2 u^1)\mathbf{v}$$

in the weak sense in $C_{\gamma(t)}$. Therefore,

$$\left(\nabla^2 u^\ell\right)\mathbf{v} = \frac{\mathbf{n}^\ell}{\mathbf{n}^1}(\nabla^2 u^1)\mathbf{v} = 0, \quad 2 \le \ell \le n+1 \text{ a.e. on } C_{\gamma(t)}.$$

Hence again by the chain rule in Lemma 3.3, $\nabla u^\ell, 2 \le \ell \le n+1$ is constant on $C_{\gamma(t)}$. Therefore, each ∇u^ℓ are constant on $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. Furthermore, since for each $1 \le \ell \le n+1$, ∇u^ℓ is continuous, we conclude that ∇u^ℓ for all $1 \le \ell \le n+1$ are constant on all $C_{\gamma(t)}$ that foliates $B^2(x_0, \delta)$. On the other hand, each 2-dimensional region U of $B^2(x_0,\delta)$ automatically satisfies all the assumptions (1) and (3) in Proposition 3.1, hence the same argument for each $C_{\gamma(t)}$ gives ∇u^ℓ for all $2 \le \ell \le n+1$ are constant on the same region on which ∇u^1 is constant. This proves ∇u is either constant on 2-dimensional regions or constant on line segments in $B^2(x_0,\delta)$ joining the boundary. The proof of Proposition 3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ follows from exactly the same argument as the last paragraph of the proof of Lemma 3.7. The proof for the base case is complete. $\square$

3.2. **Inductive step- k-dimensional slices.** In this step, we will prove that Proposition 3.1 holds true for k if it holds true for $k-1$ when $2 < k \le n$. This, combined with the base case $k = 2$ established in the previous step, completes the proof of Proposition 3.1.

3.2.1. *Developability.* Based on the induction hypothesis for $k-1$, we first prove a weaker result in k-dimensional slices of Ω than Proposition 3.1. That is, we prove that u is developable on all k-dimensional slices satisfying assumptions (1)-(4) of Proposition 3.1 in the following sense:

Proposition 3.8. *Suppose Proposition 3.1 is true for any $(k-1)$-dimensional slice of Ω on which assumptions (1)-(4) are satisfied. Let P_k be any k-dimensional plane such that assumptions (1)-(4) for u holds on $P_k \cap \Omega$, then for every $x \in \Omega$, either u is affine in a neighborhood in $P_k \cap \Omega$ of x, or there exists a unique $(k-1)$-dimensional hyperplane $P^x_{k-1} \ni x$ of P_k such that u is affine on the connected component of x in $P^x_{k-1} \cap \Omega$.*

Proof. We first need to define a terminology that is the higher dimensional version of "line segments joining the boundary of some domain at both ends".

Definition 3.9. By a *k-plane P in Σ*, we mean a connected component of a k-dimensional plane $P \cap \Sigma$, where Σ is any N-dimensional region with $N \ge k \ge 1$.

Remark 3.10. We emphasize here that such k-plane P in Σ refers to *not* the entire plane, but just the part inside a region. On the other hand, it refers to the *entire* connected part inside this region.

Let $\mathbf{v}$ be any unit directional vector of P_k, let $\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}$ be a set of *linearly independent unit* vectors of P_k perpendicular to $\mathbf{v}$, we parametrize the family of $(k-1)$-dimensional planes parallel to the space spanned by these vectors as follows:

$$P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} = \{z : z = y + \sum_{i=1}^{k-1} s_i \mathbf{v}_i,\ s_i \in \mathbb{R},\ y \in \operatorname{span}\langle \mathbf{v} \rangle\}.$$

Lemma 3.11. *Given direction $\mathbf{v}$, for a.e. $y \in \operatorname{span}\langle \mathbf{v} \rangle$, u is $C^{1,1/2}_{\text{loc}}$ and is an isometry on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. Moreover for every $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, u is either affine on a $(k-1)$-dimensional region in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ containing x, or affine on a $(k-2)$-plane in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ passing through x.*

Proof. Since u satisfies assumptions (1)-(4) on $P_k \cap \Omega$, by Fubini Theorem, for a.e. $y \in \operatorname{span}\langle \mathbf{v} \rangle$, assumptions (1)-(4) are also satisfied on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. Hence by our induction hypothesis on $(k-1)$ slices of Ω, ∇u is $C^{0,1/2}_{\text{loc}}$ on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. By assumption (2) $\nabla u^T \nabla u = \mathrm{I}$ a.e., and hence everywhere in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ by continuity. Therefore, by assumption (1) and the chain rule in Lemma 3.3, u is an isometry on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$.

Moreover by our induction hypothesis, for every $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, ∇u is either constant on a $(k-1)$-dimensional region in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ containing x, or constant on an $(k-2)$-plane in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ passing through x. Hence by the the chain rule in Lemma 3.3, u is either affine on $(k-1)$ dimensional regions in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, or affine on $(k-2)$-planes in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. The proof is complete. □

Now we want to show that a substantial part of Lemma 3.11 is true for *every* rather than a.e. $(k-1)$-dimensional planes in Ω.

Lemma 3.12. *Given direction $\mathbf{v}$, for every $y \in \operatorname{span}\langle \mathbf{v} \rangle$ and for every $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, u is either an affine isometry on a $(k-1)$-dimensional region in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ containing x, or an affine isometry on a $(k-2)$-plane in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ passing through x.*

Remark 3.13. We obtain from the proof of Lemma 3.11 that u is C^1 on a.e. planes. However, Lemma 3.11 does *not* imply u is C^1 on *every* plane because even though ∇u is continuous on a.e. planes, we cannot conclude from here that ∇u is continuous in Ω, so we cannot pass to the limit.

Proof. Given $y \in \mathrm{span}\langle \mathbf{v} \rangle$, Lemma 3.11 guarantees a sequence $y_m \in \mathrm{span}\langle \mathbf{v} \rangle$, $y_m \to y$ such that Lemma 3.11 is true on $P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ for every m.

Let $x \in P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, we divide the proof into the following two cases:

(1) There is a sequence of $(k-2)$-planes P_m in $P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ on which u is an affine isometry and P_m converges to x in distance.
(2) There does not exist such a sequence of $(k-2)$-planes.

Suppose we are in case (1), then the limit of P_m must also be a $(k-2)$-plane P in $P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ passing through x. Also since u is Lipschitz continuous, u must also be an affine isometry on P, which proves the Lemma in this case (Figure 6).

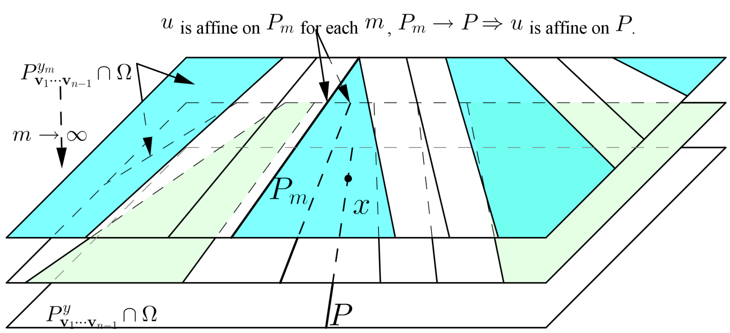

FIGURE 6. Case (1).

Suppose now we are in case (2). If we cannot find such a sequence of $(k-2)$-planes, then we must find $x_m \in P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, $x_m \to x$ with the property that there is $\epsilon > 0$ such that u is an affine isometry on $B^{k-1}(x_m, \epsilon) \subset P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$. Otherwise, there will again be a sequence of $(k-2)$-planes (i.e. the boundaries of the maximal affine regions containing x_m) converging to x in distance, contradiction to the fact that we are in case (2). Continuity of u then must force u to be an affine isometry on $B^{k-1}(x, \epsilon) \subset P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{n-1}} \cap \Omega$, which again proves the lemma in this case (Figure 7). The proof is complete. □

Lemma 3.14. *Suppose u is an affine isometry on two line segments C_1 and C_2 in $P_k \cap \Omega$ intersecting at a point x in the interior of both C_1 and C_2. Let H be the convex hull of the line segments C_1 and C_2, then u is an affine isometry on $H \cap \Omega$.*

Proof. We parametrize C_1 and C_2 by $\{x + t\mathbf{v}_1, t \in [-a, b]\}$ and $\{x + s\mathbf{v}_2, s \in [-c, d]\}$, respectively, with both $\mathbf{v}_1$ and $\mathbf{v}_2$ unit vectors. We can assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, otherwise,

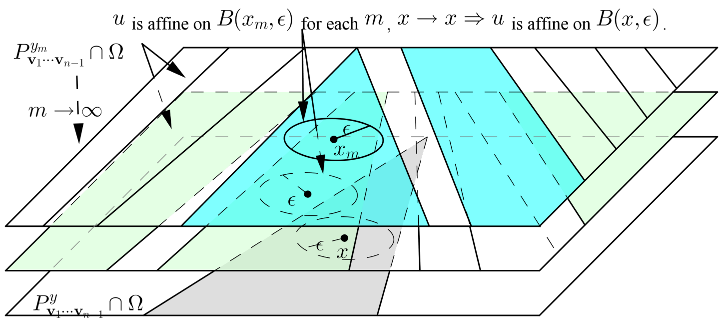

FIGURE 7. Case (2)

the conclusion of the lemma is obvious. Since u is affine on both C_1 and C_2, $u(C_1)$ and $u(C_2)$ are both line segments in $\mathbb{R}^{n+1}$. We can again parametrize the lines that contains the line segments $u(C_1)$ and $u(C_2)$ by $u(x)+t\tilde{\mathbf{v}}_1$ and $u(x)+s\tilde{\mathbf{v}}_2$, both $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ are unit vectors due to the isometry assumption.

Let $y \in H \cap \Omega$, we can of course assume that y is neither in C_1 nor C_2, otherwise, there is nothing to prove. In this way, we can find a line L_3 passing through y and intersect C_1 at only one point, denoted x_{13}; and C_2 at only one point, denoted x_{23} and the segment $\overline{x_{13}x_{23}}$ lies inside Ω. Since $x_{13} \in C_1$, $x_{13} = x + t_0\mathbf{v}_1$ for some $t_0 \in [-a, b]$. Similarly $x_{23} = x + s_0\mathbf{v}_2$ for some $s_0 \in [-c, d]$. Then since

$$y = wx_{13} + (1-w)x_{23} \quad \text{for some } w \in [0,1], \tag{3.4}$$

It follows

$$y = x + wt_0\mathbf{v}_1 + (1-w)s_0\mathbf{v}_2.$$

To prove that u is an affine isometry on H, we need to prove

$$u(y) = u(x) + wt_0\tilde{\mathbf{v}}_1 + (1-w)s_0\tilde{\mathbf{v}}_2. \tag{3.5}$$

We first claim that the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2. Since x is in the interior of C_1 and C_2, we can construct a parallelogram $ABCD$ centered at x, with $A, C \in C_1$ and $B, D \in C_2$. Since u is an affine isometry on C_1 and C_2, $|u(A)-u(x)| = |A-x|$, $|u(B)-u(x)| = |B-x|$, $|u(C)-u(x)| = |C-x|$ and $|u(D)-u(x)| = |D-x|$. On the other hand, $|u(A)-u(B)| \le |A-B|$ and $|u(B)-u(C)| \le |B-C|$ since u is 1-Lipschitz (Figure 8).

This implies the angle α_2 between the line segments $\overline{u(x)u(A)}$ and $\overline{u(x)u(B)}$ must be smaller than or equal to the angle α_1 between $\overline{xA}$ and $\overline{xB}$, and the angle β_2 between the line segments

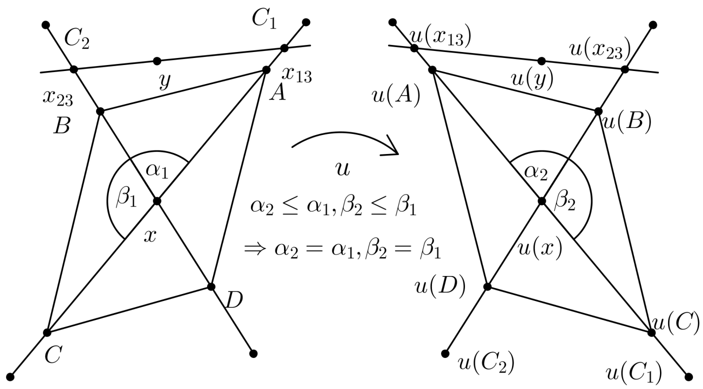

FIGURE 8.

$\overline{u(x)u(B)}$ and $\overline{u(x)u(C)}$ must be smaller than or equal to the angle β_1 between $\overline{xB}$ and $\overline{xC}$. Hence $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. This proves our claim.

Since by assumption, u is an affine isometry on $\overline{x_{13}x}$ and $\overline{x_{23}x}$, we have

$$u(x_{13}) - u(x) = t_0\tilde{\mathbf{v}}_1 \quad \text{ and } \quad u(x_{23}) - u(x) = s_0\tilde{\mathbf{v}}_2.$$

for the same t_0, s_0 and unit vector $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$ as defined before. In particular, $|u(x_{13})-u(x)| = |x_{13}-x|$ and $|u(x_{23}) - u(x)| = |x_{23} - x|$. Moreover, since the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2, we have $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$.

On the other hand, $u(\overline{x_{13}x_{23}})$ is a 1-Lipschitz curve, hence the length the the curve $u(\overline{x_{13}x_{23}})$, denoted by $|u(\overline{x_{13}x_{23}})|$, satisfies $|u(\overline{x_{13}x_{23}})| \leq |x_{13} - x_{23}|$. Altogether we have

$$|u(x_{13}) - u(x_{23})| \leq |u(\overline{x_{13}x_{23}})| \leq |x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|.$$

This implies

$$|u(\overline{x_{13}x_{23}})| = |u(x_{13}) - u(x_{23})|.$$

Hence the curve $u(\overline{x_{13}x_{23}})$ must coincide with line segment $\overline{u(x_{13})u(x_{23})}$. Therefore, u also maps the line segment $\overline{x_{13}x_{23}}$ onto a line segment $\overline{u(x_{13})u(x_{23})}$, which means u is affine on $\overline{x_{13}x_{23}}$.

Finally, since u is 1-Lipschitz, $|u(x_{13})-u(y)| \leq |x_{13}-y|$ and $|u(x_{23})-u(y)| \leq |x_{23}-y|$. However, since u is affine on $\overline{x_{13}x_{23}}$,

$$|u(x_{13}) - u(x_{23})| = |u(x_{13}) - u(y)| + |u(y) - u(x_{23})| \leq |x_{13} - y| + |y - x_{23}| = |x_{13} - x_{23}|.$$

But we already showed that $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$. Hence $|u(x_{13}) - u(y)| = |x_{13} - y|$ and $|u(x_{23}) - u(y)| = |x_{23} - y|$. Therefore,

$$u(y) = wu(x_{13}) + (1 - w)u(x_{23})$$

for the same w as (3.4), which yields (3.5). The proof is complete. □

Corollary 3.15. *Given a ℓ-dimensional ($\ell \leq k$) neighborhood U in $P_k \cap \Omega$, and a line segment C in $P_k \cap \Omega$ such that $x \in C \cap U$ is in the interior of both U and C, if u is an affine isometry on both U and C, then u is an affine isometry on the convex hull H of U and C inside Ω*

Proof. Let $y \in H \cap \Omega$. We need to show that $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some $\tilde{\mathbf{v}}$ as a linear combination of directional vectors in $u(U)$ and $u(C)$ and $|t\tilde{\mathbf{v}}| = |y - x|$. Let P_y be a 2-dimensional plane that contains y and C. Then P_y intersects U at some line segment C_y. Since u is an affine isometry on both C and C_y, by Lemma 3.14, u is an affine isometry on the convex hull of C and C_y (Figure 9). Since this convex hull contains both y and x, this implies $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some vector $\tilde{\mathbf{v}}$,

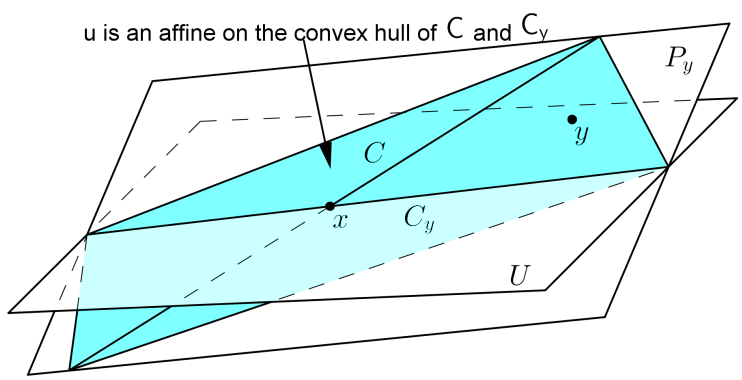

FIGURE 9.

$|t\tilde{\mathbf{v}}| = |y - x|$, and $\tilde{\mathbf{v}}$ is a linear combination of directional vectors of $u(C)$ and $u(C_y)$. Our claim then follows because $C_y \subset U$ and u is an affine isometry on U, so any vectors of $u(C_y)$ is a linear combination of vectors in $u(U)$. The proof is complete. □

By obvious induction we then have

Corollary 3.16. *Suppose U_1 and U_2 are k_1 and k_2-dimensional neighborhoods ($k_1, k_2 \leq k$) in $P_k \cap \Omega$ with nonempty intersections. Moreover, there exists a point $x \in U_1 \cap U_2$ belonging to the interior of both U_1 and U_2. If u is an affine isometry on both U_1 and U_2, then u is an affine isometry on the convex hull of U_1 and U_2 inside Ω.*

Now we are ready to prove Proposition 3.8. Given $x \in P_k \cap \Omega$, we first claim that there is a $(k-1)$-dimensional hyperplane P_0^x in P_k and a $(k-1)$-dimensional isometric affine neighborhood $U_0^x \subset P_0^x \cap \Omega$ such that $x \in U_0^x$. Otherwise, for all $(k-1)$-dimensional hyperplanes in $P_k \cap \Omega$ that

pass through x, x is not contained in any $(k-1)$-dimensional affine neighborhood. In particular, let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be linearly independent vectors of P_k and Let $P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{v}_k}$, $i = 1, \cdots, n$ be the $(k-1)$-dimensional hyperplanes in Ω passing through x and parallel to the space spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_k$. Then x is not contained in any $(k-1)$-dimensional affine neighborhood in $P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{v}_k} \cap \Omega$. Thus, by Lemma 3.12 there exists $(k-2)$-planes $P_{\hat{i}} \ni x$ in $P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{v}_k} \cap \Omega$ and u is an affine isometry on $P_{\hat{i}}$ for each i. By Corollary 3.16, u is an affine isometry on the convex hull of $P_{\hat{i}}$ for all $1 \le i \le k$ (Figure 10 Case 1). Let $\mathbf{v}_{\hat{i}}$ be a directional vector of $P_{\hat{i}}$. Since $P_{\hat{i}} \subset P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{e}_k}$, which is orthogonal to $\mathbf{v}_i$, at least $k-1$ out of these k vectors are linearly independent. This convex hull has $k-1$ linearly independent directional vectors, hence it must be a $(k-1)$-dimensional neighborhood, contradiction to our assumption, which proves our claim.

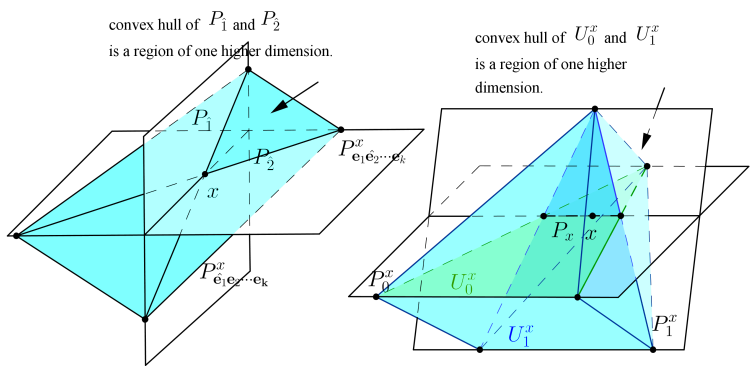

FIGURE 10. Case 1 (left) and Case 2 (right).

Therefore, x must be contained in a $(k-1)$-dimensional isometric affine neighborhood $U_0^x \subset P_0^x \cap \Omega$ for some $(k-1)$-dimensional hyperplane P_0^x. If U_0^x is the entire connected component containing x in $P_0^x \cap \Omega$, then the conclusion of the proposition is achieved. Otherwise, we can find a maximal $(k-2)$-plane P_x in U_0^x, which is *not* a $(k-2)$-plane in $P_0^x \cap \Omega$, i.e., it is away from $\partial\Omega$, on which u is an affine isometry. Let P_1^x be any other $(k-1)$-dimensional hyperplane containing the region P_x. We have $P_x = U_0^x \cap P_1^x$ and since the maximal affine region $P_x \subset P_1^x \cap \Omega$ is not a $(k-2)$-plane in $P_1^x \cap \Omega$, by Lemma 3.12, x must be contained in a $(k-1)$-dimensional neighborhood $U_1^x \subset P_1^x \cap \Omega$ on which u is an affine isometry (Figure 10 Case 2). By Corollary 3.16, u is affine on the convex hull of U_0^x and U_1^x, whose interior is a k-dimensional neighborhood, which also achieves the conclusion of Proposition 3.8. The proof is complete. □

3.2.2. *Regularity and the conclusion of the inductive step.* In out last step, we will essentially show that the conclusion of Proposition 3.8 combined with assumptions (1)-(4) of Proposition 3.1 for a k-dimensional slice P_k, implies the conclusion of the latter proposition. This will hence conclude

the inductive step. The key is to show that if u is affine on a $(k-1)$-plane, then its full gradient must be constant on the same region. The arguments are very similar to what we used in the proofs of Lemmas 3.5-3.7. We will first prove the following lemma.

Lemma 3.17. *Suppose on a k-plane P $(1 \le k \le n)$ in Ω we have the following:*

(1) *There is a sequence of smooth functions $u^\epsilon \in C^\infty(\Omega, \mathbb{R}^{n+1})$ such that*
$$\int_P |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^k \to 0.$$
(2) $\operatorname{rank} \nabla^2 u^\ell \le 1$ *and $\nabla^2 u^\ell$ is symmetric a.e. on P for all $1 \le \ell \le n+1$.*

Then if u is affine on P, ∇u is constant on P.

Proof. Let $\mathbf{v}$ be any unit directional vector in P. By assumption (1) and the chain rule in Lemma 3.3, u is affine on P implies
$$\nabla u(x)\mathbf{v} = \text{constant} \quad \text{for a.e. } x \in P.$$
Take the direction derivative one more time, together with assumption (1) we obtain,
$$(\mathbf{v})^T \nabla^2 u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{3.6}$$
for all $1 \le \ell \le n+1$. However, to show that ∇u is constant on P, we need a conclusion stronger than (3.6), i.e.,
$$\nabla^2 u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{3.7}$$
for all $1 \le \ell \le n+1$. We will show that under our assumptions, (3.6) implies (3.7). Indeed, By assumption (2), we can write $\nabla^2 u^\ell$ as
$$\nabla^2 u^\ell(x) = \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x) \quad \text{a.e.}$$
for some scalar function λ and $\mathbf{b} \in \mathbb{S}^{n-1}$. Then (3.6) implies,
$$(\mathbf{v})^T \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x)\mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle^2 = 0 \quad \text{a.e.}$$
This then implies
$$\lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle = 0 \quad \text{a.e.}$$
Therefore,
$$\nabla^2 u^\ell \mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle \mathbf{b}(x) = 0 \quad \text{a.e.}$$
which is exactly (3.7). The proof of the lemma is complete. □

Let P_k be any k-dimensional plane such that assumptions (1)-(4) in Proposition 3.1 for u holds on $P_k \cap \Omega$. Suppose u is affine on some maximal neighborhood $U \subset P_k \cap \Omega$, by continuity of u,

it is also affine on its closure $\overline{U} \cap \Omega$. Now if $x \in \partial U \cap \Omega$, then x is not contained in an affine neighborhood of u, therefore by Proposition 3.8, it is affine on a unique $(k-1)$-plane P_x^U in $P_k \cap \Omega$ passing through x, which implies $\partial U \cap \Omega \subset \bigcup_{x\in\partial U\cap\Omega} P_x^U$. On the other hand, suppose u is affine on some $(k-1)$-plane P_x^U in $P_k \cap \Omega$ passing through $x \in \partial U \cap \Omega$. Since u is affine on U and P_x^U, which intersect at x, it must be affine on the convex hull of U and P_x^U inside Ω by Corollary 3.16. But U is maximal, hence $\bigcup_{x\in\partial U\cap\Omega} P_x^U \subset \partial U \cap \Omega$. Therefore,

$$\partial U \cap \Omega = \bigcup_{x\in\partial U\cap\Omega} P_x^U.$$

Moreover, Corollary 3.16 ensures for $x, z \in \partial U \cap \Omega$, $P_x^U = P_z^U$ if $z \in P_x^U$ and $P_x^U \cap P_z^U \cap \Omega = \emptyset$ if $z \notin P_x^U$.

Similar as in the proof of Lemma 3.7 (Figure 5), it suffices to show that the conclusions hold locally true. If $x_0 \in P_k \cap \Omega$ is a point lying in an affine neighborhood for u in P_k, then Lemma 3.17 and the assumptions of Proposition 3.1 immediately imply that ∇u must be constant in the same neighborhood, which is the desired conclusion. Otherwise, we claim we can choose small enough $\delta > 0$ so that for any region U on which u is affine, the k-dimensional ball $B^k(x_0, \delta) \subset P_k \cap \Omega$ intersects ∂U at no more than *two* $(k-1)$-planes belonging to ∂U. Indeed, since for any maximal constant region U, no $(k-1)$-planes in ∂U intersect inside Ω, for any possible sequence of maximal affine regions U_m converging to x_0 in distance, there are at most two $(k-1)$-planes $P_{x_1}^{U_m}$ and $P_{x_2}^{U_m}$ on ∂U_m, converging to a $(k-1)$-plane passing through x, whose angle (if they are nonparallel) or distance (if they are parallel) converges to zero. As a consequence, all the other $(k-1)$-planes on ∂U_m must be arbitrarily close to $\partial\Omega$, and we can choose δ small enough so that $B^k(x_0, \delta)$ is away from $\partial\Omega$ and hence it does not intersect a third $(k-1)$-planes on ∂U_m (Figure 2).

We now focus on $B^k(x_0, \delta) \subset P_k \cap \Omega$. For any $x \in B^k(x_0, \delta)$, we want to construct a $(k-1)$-plane P_x in $B^k(x_0, \delta)$ passing through x on which u is affine and $P_x \cap P_z \cap B^k(x_0, \delta) = \emptyset$ if $z \notin P_x$. For those x not contained in an affine region of u, this $(k-1)$-plane is given automatically by Proposition3.8 and Corollary 3.16. If x is contained in an affine maximal region U of u, then it is affine on every $(k-1)$-planes in U that passes through it so we have to choose the appropriate one: 1) If $B^k(x_0, \delta)$ intersect only one $(k-1)$-plane P^U in $P_k \cap \Omega$ that belongs to ∂U, then we define P_x to be the $(k-1)$-plane in $B^k(x_0, \delta)$ passing through x and parallel to P^U; 2) If $B^k(x_0, \delta)$ intersects two $(k-1)$-planes P_1^U, P_2^U in $P_k \cap \Omega$ that belongs to ∂U, let P_1 and P_2 be the two $(k-1)$-dimensional hyperplane that contain P_1^U and P_2^U. If P_1 and P_2 are not parallel, let $O := P_1 \cap P_2$ and let P_x be the $(k-1)$-plane in $B^k(x_0, \delta)$ passing through x whose extension goes through O. If P_1 and P_2 are parallel, then we let P_x be the $(k-1)$-plane $B^k(x_0, \delta)$ inside $B^k(x_0, \delta)$ passing through x and parallel to P_1. (Figure 3).

In this way, we construct a family of $(k-1)$-planes $\{P_x\}_{x\in B^k(x_0,\delta)}$ in $B^k(x_0,\delta)$ on which u is affine and $P_x \cap P_z \cap B^k(x_0,\delta) = \emptyset$ if $z \notin P_x$. For every $x \in B^k(x_0,\delta)$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector in $B^k(x_0,\delta)$ orthogonal to P_x. By making δ smaller we can make sure that none of the P_xs intersect inside $B^k(x_0,2\delta)$, and therefore we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector field inside the ball of radius δ. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

then has a unique solution $\gamma : (a,b) \to B^k(x_0,\delta)$ for some interval $(a,b) \in \mathbb{R}$ containing 0. Moreover, if necessary by choosing a smaller δ we can make sure that $\cup\{P_{\gamma(t)}\}_{t\in(a,b)} = B^k(x_0,\delta)$. Therefore, $\{P_{\gamma(t)}\}_{t\in(a,b)}$ is a foliation of $B^k(x_0,\delta)$ (Figure 4).

We define the function $h : B^k(x_0,\delta) \to B^k(x_0,\delta)$ as

$$h(x) = \gamma(t) \quad \text{if } x \in P_{\gamma(t)}.$$

Since none of the $P_{\gamma(t)}$ intersect inside $B^k(x_0,\delta)$, h is well defined and h is constant along each $P_{\gamma(t)}$, i.e., $h^{-1}(\gamma(t)) = P_{\gamma(t)}$. Since γ is Lipschitz, h is Lipschitz as well. Moreover, since $|\gamma''(t)|$ is uniformly bounded, we have the Jacobian $J_h > C > 0$.

We now want to show the assumptions of Lemma 3.17 are satisfied along $P_{\gamma(t)}$ for a.e. $t \in (a,b)$. Let E_0 be the set of all $x \in B^k(x_0,\delta)$ such that rank $\nabla^2 u^\ell(x) > 1$ or $\nabla^2 u^\ell(x)$ is not symmetric for any $1 \le \ell \le n+1$. By our assumptions in Proposition 3.1 on u, $|E_0| = 0$. As h is Lipschitz, we can apply the co-area formula to h to obtain,

$$\begin{aligned} 0 = \int_{E_0} J_h(x)\,dx \;&= \int_\gamma \mathcal{H}^{k-1}\big(E_0 \cap h^{-1}(w)\big)d\mathcal{H}^1(w) \;&= \int_a^b \mathcal{H}^{k-1}\big(E_0 \cap h^{-1}((\gamma(t))\big)|\gamma'(t)|\,dt \\ & &= \int_a^b \mathcal{H}^{k-1}\big(E_0 \cap P_{\gamma(t)}\big)|\gamma'(t)|\,dt. \end{aligned}$$

Therefore, for a.e. $t \in (a,b)$, $\mathcal{H}^{k-1}\big(E_0 \cap P_{\gamma(t)}\big) = 0$ since $|\gamma'| = 1$. Moreover, by change of variable formula,

$$\begin{aligned} \int_{B^k(x_0,\delta)} & \big(|u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2\big) J_h \\ &= \int_\gamma \int_{h^{-1}(w)} |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^{k-1} d\mathcal{H}^1(w) \\ &= \int_a^b \int_{h^{-1}(\gamma(t))} |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^{k-1} |\gamma'(t)|\,dt \\ &= \int_a^b \int_{P_{\gamma(t)}} |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^{k-1} |\gamma'(t)|\,dt. \end{aligned}$$

Since J_h is bounded, together with assumption (1) in Proposition 3.1, we then have for a.e. $t \in (a, b)$,

$$\int_{P_{\gamma(t)}} |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^{k-1} \to 0.$$

Therefore, the assumptions of Lemma 3.17 are satisfied along $P_{\gamma(t)}$ for a.e. $t \in (a, b)$. It follows that ∇u is constant on $P_{\gamma(t)}$ for a.e. $t \in (a, b)$.

By choosing an initial value for γ arbitrary close to x_0 and applying the co-area formula in a similar manner we can make sure that ∇u is of class $W^{1,2}$ on γ. Hence we conclude that ∇u is $C^{0,1/2}$ on γ by the Sobolev embedding theorem. Let F be the set of $t \in (a, b)$ such that ∇u is not constant along $P_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify ∇u to be constant along $P_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^k(\bigcup\{P_{\gamma(t)} : t \in F\}) \leq (2\delta)^{k-1}(\sup J_h^{-1})\mathcal{H}^1(\{\gamma(t) : t \in F\}) = (2\delta)^{k-1}(\sup J_h^{-1})\mathcal{H}^1(F) = 0.$$

Hence ∇u is $C^{0,1/2}$ up to modification of a set of measure zero in $B^k(x_0, \delta)$. Moreover, ∇u is constant on $P_{\gamma(t)}$ for all t, which foliates $B^k(x_0, \delta)$. Thus ∇u is constant on any region on which u is affine. Therefore, ∇u is either constant on a $(k-1)$-plane or k-dimensional region in $B^k(x_0, \delta)$. This implies that the conclusions of Proposition 3.1 under the induction hypothesis are true and hence the inductive step is established. As a conclusion the proofs of Proposition 3.1 and Theorem 1.4 are complete. □

4. Density: Proof of Theorem 1.5

In this section we show isometric immersions smooth up to the boundary are strongly dense in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ if $\Omega \subset \mathbb{R}^n$ is a convex C^1 domain. Note that it is sufficient to prove that $I^{2,2} \cap C^\infty(\Omega, \mathbb{R}^{n+1})$ is strongly dense in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$. Having this result at hand, and since Ω is assumed convex, the approximating sequence can be easily rescaled to be smooth up to the boundary.

4.1. **Foliations of the domain.** We have argued in the proof of Theorem 1.4 in section 3.2.2 that for every maximal region $U \subset \Omega$ on which u is affine, $\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U$, where P_x^U is some $(n-1)$-plane in Ω containing x with the property that for $x_1, x_2 \in \partial U \cap \Omega$, $P_{x_1}^U = P_{x_2}^U$ if $x_2 \in P_{x_1}^U$ and $P_{x_1}^U \cap P_{x_2}^U \cap \Omega = \emptyset$ if $x_2 \notin P_{x_1}^U$.

We say a maximal region on which u is affine is a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω.

Lemma 4.1. *It is sufficient to prove Theorem 1.5 for a function in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ with finite number of bodies.*

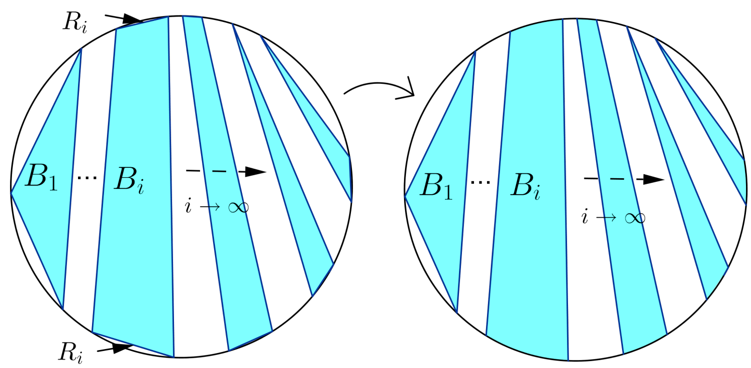

FIGURE 11.

Proof. We will show that we can approximate a function $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ by maps in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ with finite number of bodies. First we can assume that all bodies are pairwise disjoint, otherwise, they must be contained in one body since ∇u is continuous and constant on all bodies. There can be at most countably many such disjoint bodies and so we label them B_i for $i \in \mathbb{N}$. As Ω is bounded, $\sum |B_i| < \infty$. Given B_i for i large enough, there must be two $(n-1)$-planes $P_1^{B_i}$ and $P_2^{B_i}$ in ∂B_i such that the angle between them is arbitrarily small if they are nonparallel, or their distance is arbitrarily small if they are parallel. We denote by U_i the regions bounded by $P_1^{B_i}$, $P_2^{B_i}$ and $\partial\Omega$. It can be shown that given $\epsilon > 0$, there is M large enough for which $\sum_{i \geq M} |U_i| < \epsilon$. Let $R_i := U_i \setminus B_i$ (Figure 11 left).

It easily follows that $\sum_{i \geq M} |R_i| < \epsilon$. Then for every $i \geq M$, since u is affine on B_i, we can modify u to u^ϵ by affine extension to R_i (Figure 11 right). Obviously the new map u^ϵ satisfies $u^\epsilon \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ with $\|u^\epsilon\|_{W^{2,2}} \leq C\|u\|_{W^{2,2}}$ for a universal constant $C > 0$ depending on the dimensions of Ω. Moreover $\nabla u^\epsilon \in O(n, n+1)$ a.e., which means $u^\epsilon \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$. Since $\sum_{i \geq M} |R_i| < \epsilon$ which is arbitrarily small, $\|u^\epsilon - u\|_{W^{2,2}}$ is arbitrarily small by absolute continuity of integrable functions. The proof is complete. $\square$

Now we can just assume $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ has finite number of bodies. Each body is closed and so is therefore their union, whose complement we denote by $\widetilde{\Omega}$. Note that now for every n-dimensional maximal-affine region $U \subset \widetilde{\Omega}$, $\partial U \cap \widetilde{\Omega}$ consists of at most two $(n-1)$-planes.

Similarly as in the proof of Lemma 3.7, for every $x \in \widetilde{\Omega}$, we will construct an $(n-1)$-plane P_x in $\widetilde{\Omega}$ passing through it on which ∇u is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$. For those x not contained in a constant region of ∇u, this $(n-1)$-plane in $\widetilde{\Omega}$ is given automatically by Theorem 1.4. If x is contained in a constant maximal region U of ∇u, then it is constant on every $(n-1)$-plane in U that passes through it so we have to choose the appropriate one: 1) If $\partial U \cap \widetilde{\Omega}$

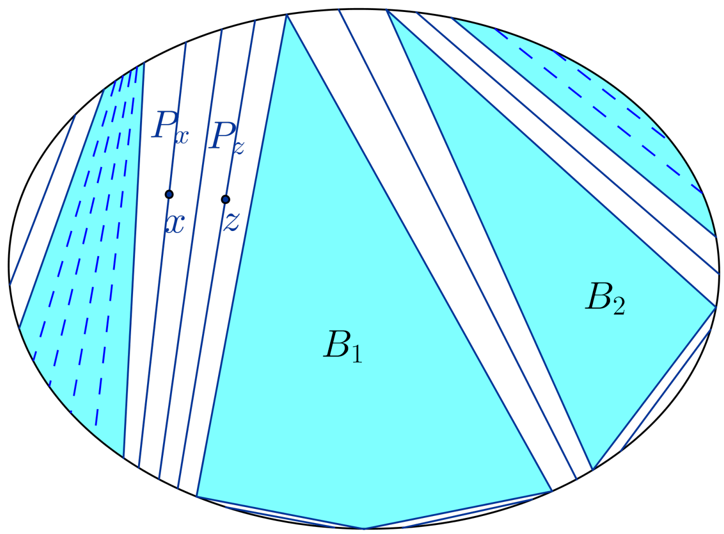

FIGURE 12. Construction of global foliations in $\widetilde{\Omega}$.

consists of only one $(n-1)$-plane P^U in $\widetilde{\Omega}$, we define P_x to be the $(n-1)$-plane in $\widetilde{\Omega}$ passing through x and parallel to P^U; 2) If $\partial U \cap \widetilde{\Omega}$ consists of two $(n-1)$-planes P_1^U, P_2^U in $\widetilde{\Omega}$, let P_1 and P_2 be the two $(n-1)$ dimensional hyperplanes that contain P_1^U and P_2^U. If P_1 and P_2 are not parallel, let $A := P_1 \cap P_2$ and let P_x be the $(n-2)$-plane in $\widetilde{\Omega}$ passing through A and x. If P_1 and P_2 are parallel, then we let P_x be the $(n-2)$-plane passing through x and parallel to P_1. The component $P_x \cap \Omega$ is then our desired $(n-1)$-plane in Ω (and we still denote it P_x). In this way, we construct a family of $(n-1)$-plane $\{P_x\}_{x\in\widetilde{\Omega}}$ in $\widetilde{\Omega}$ on which ∇u is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$ (Figure 12).

For every $x \in \widetilde{\Omega}$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector orthogonal to the family P_x constructed above. Since none of the P_xs intersect inside $\widetilde{\Omega}$ we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector fields. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0$$

has a unique solution $\gamma : (a, b) \to \widetilde{\Omega}$ for some interval $(a, b) \subset \mathbb{R}$ containing 0. Note that $P_x = P_{\gamma(t)}$ if $x \in P_{\gamma(t)}$, therefore, $\{P_{\gamma(t)}\}_{t\in(a,b)}$ is a local foliation of $\widetilde{\Omega}$ such that ∇u is constant on $P_{\gamma(t)}$ for all $t \in (a, b)$ (Figure 13).

4.2. **Leading curves in the domain.**

Definition 4.2. Let $\{P_x\}_{x\in\widetilde{\Omega}}$ be a family of $(n-1)$-planes in $\widetilde{\Omega}$ passing through x on which ∇u is constant, satisfying $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$ and $P_x = P_z$ if $z \in P_x$. We say that a curve

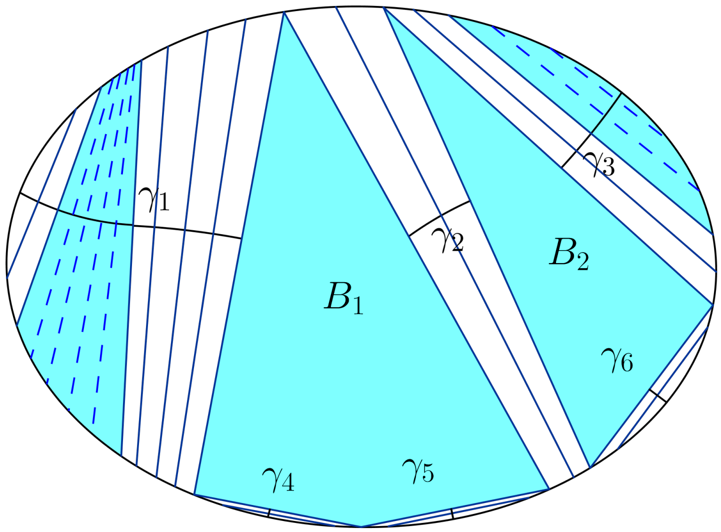

FIGURE 13.

$\gamma \in C^{1,1}([0,\ell],\widetilde{\Omega})$ parametrized by arclength is a leading curve if it is orthogonal at any possible point of intersection $z \in \gamma([0,\ell]) \cap P_x$ to $P_x = P_z$ for all $x \in \widetilde{\Omega}$ (Figure 14).

It is easy to see that γ constructed in Subsection 4.1 when restricted to the interval $[0,\ell]$ is a Leading curve, since by the ODE

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

$|\gamma'| = 1$ and $|\gamma''|$ is bounded as $\mathbf{N}$ is Lipschitz.

Definition 4.3. The $(n-1)$-dimensional hyperplane $F_\gamma(t)$ orthogonal to $\gamma(t)$ at $t \in [0,\ell]$ is called the Leading front of γ at $t \in [0,\ell]$ (Figure 14).

Remark 4.4. It then follows from the definition of the Leading curve that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \widetilde{\Omega} = \emptyset$ for all $t,\tilde{t} \in [0,\ell]$ such that $t \neq \tilde{t}$. Moreover, $F_\gamma(t) \cap \widetilde{\Omega} = F_\gamma(t) \cap \Omega$, otherwise, $F_\gamma(t) \cap B \neq \emptyset$ where B is one of the bodies in $\Omega \setminus \widetilde{\Omega}$. Since ∇u, being continuous, is constant on $F_\gamma(t) \cap \widetilde{\Omega}$ and B, it must be constant on their convex hull, which is again a body, contradiction to that a body is a maximal region. Therefore, $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = \emptyset$ for all $t, \tilde{t} \in [0,\ell], t \neq \tilde{t}$.

We say that a curve γ covers the domain $A \subset \Omega$ if

$$A \subset \bigcup \{F_\gamma(t) : t \in [0,\ell]\}.$$

By $\Omega(\gamma)$ we refer to the biggest set covered by γ in Ω. We now restrict our attention to the covered domain $\Omega(\gamma)$. It is obvious that $\Omega(\gamma)$ is convex since it is bounded by $F_\gamma(0)$, $F_\gamma(\ell)$ and $\partial\Omega$.

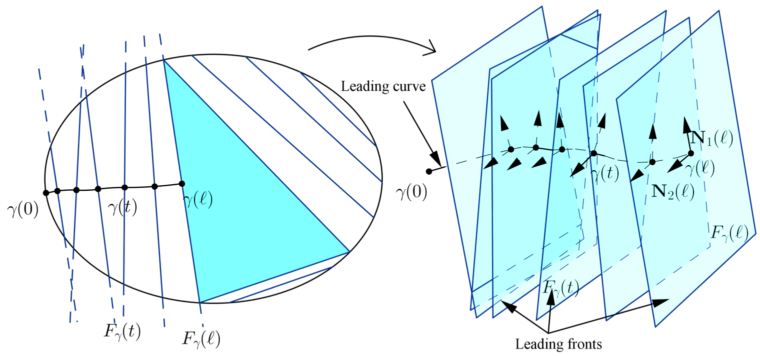

FIGURE 14. Leading curve and Leading fronts.

From the construction in Subsection 4.1, the $(n-1)$-planes $P_{\gamma(t)}$ in $\widetilde{\Omega}$, $t \in [0,\ell]$ which constitute a local foliation of $\widetilde{\Omega}$ are global foliations of $\Omega(\gamma)$. Moreover, $P_{\gamma(t)} = F_\gamma(t) \cap \Omega(\gamma) = F_\gamma(t) \cap \Omega$ for all $t \in [0,\ell]$. We relabel them $P_\gamma(t)$ to be in consistence of notation and we name them:

Definition 4.5. The component $P_\gamma(t) := F_\gamma(t) \cap \Omega$ is called the Leading $(n-1)$-planes in Ω of γ at $t \in [0,\ell]$.

Let $\mathbf{N}_1(t), \mathbf{N}_2(t), \cdots, \mathbf{N}_{n-1}(t)$ be an orthonormal basis for the Leading front $F_\gamma(t)$ (Figure 14) such that $\mathbf{N}_i$ is Lipschitz for all $1 \leq i \leq n-1$ and $\det[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \gamma'(t)] = 1$. It is obvious such orthonormal basis exists because we can pick $\mathbf{N}_1(0), \mathbf{N}_2(0), \cdots, \mathbf{N}_{n-1}(0)$ as an orthonormal basis for $F_\gamma(0)$ that form a positive orientation with $\gamma'(0)$ and then move this frame along γ in an orientation preserving way (note that γ is not a closed curve so this is possible). Let $\Phi : [0,\ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as,

$$\Phi(t,s) := \gamma(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i(t), \tag{4.1}$$

where $s = (s_1, \cdots, s_{n-1})$. Then we can represent the Leading front at $t \in [0,\ell]$ as,

$$F_\gamma(t) = \{\Phi(t,s), s = (s_1, \cdots, s_{n-1}) \in \mathbb{R}^{n-1}\}. \tag{4.2}$$

For each $t \in [0,\ell]$, define the open set,

$$\Sigma^\gamma(t) = \{s = (s_1, \cdots, s_{n-1}) \in \mathbb{R}^{n-1} : \Phi(t,s) \in \Omega\}. \tag{4.3}$$

It is obvious that $0 \in \Sigma^\gamma(t)$, hence it is non-empty open subset of $\mathbb{R}^{n-1}$. Then we can also parametrize the Leading planes as

$$P_\gamma(t) = \{\Phi(t,s), s = (s_1, \cdots, s_{n-1}) \in \Sigma^\gamma(t)\}. \tag{4.4}$$

Now define,

$$\Sigma^\gamma := \{(t, s), \Phi(t, s) \in \Omega\}. \tag{4.5}$$

Of course we can also write,

$$\Sigma^\gamma = \{(t, s), t \in [0, \ell], s = (s_1, \cdots, s_{n-1}) \in \Sigma^\gamma(t)\}.$$

We will focus on the restriction of Φ in Σ^γ. However, if no confusion is caused, we still denote such restriction Φ. It is easy to see Φ maps Σ^γ into $\Omega(\gamma)$. Indeed, if $x = \Phi(t, s)$ for some $(t, s) \in \Sigma^\gamma$, by definition of Σ^γ, $\Phi(t, s) \in \Omega$. On the other hand, $\Phi(t, s) \in F_\gamma(t)$, thus, $x = \Phi(t, s) \in F_\gamma(t) \cap \Omega \subset \Omega(\gamma)$.

Lemma 4.6. $\Phi : \Sigma^\gamma \to \Omega(\gamma)$ *is one-to-one and onto. In particular,*

$$\Omega(\gamma) = \{\Phi(t, s), (t, s) \in \Sigma^\gamma\} = \bigcup \{P_\gamma(t) : t \in [0, \ell]\}.$$

Proof. We first show one-to-one. Suppose $\Phi(t_1, s_1) = \Phi(t_2, s_2)$ while $(t_1, s_1) \neq (t_2, s_2)$. Since $s \to \Phi(t, s)$ is obviously one-to-one by the definition of Φ, it must be $t_1 \neq t_2$. We have argued in Remark 4.4 that $F_\gamma(t_1) \cap F_\gamma(t_2) \cap \Omega = \emptyset$. Therefore, $F_\gamma(t_1) \cap F_\gamma(t_2) \cap \Omega(\gamma) = \emptyset$ since $\Omega(\gamma) \subset \Omega$. However, $\Phi(t_1, s_1) \in F_\gamma(t_1)$ and $\Phi(t_2, s_2) \in F_\gamma(t_2)$, contradiction to $\Phi(t_1, s_1) = \Phi(t_2, s_2)$.

We will now show onto. Let $x \in \Omega(\gamma)$, then $x = \Phi(t, s)$ for some $t \in [0, \ell]$ and $s \in \mathbb{R}^{n-1}$. Since $x \in \Omega(\gamma)$, $\Phi(t, s) \in \Omega(\gamma) \subset \Omega$, hence $(t, s) \in \Sigma^\gamma$. The proof is complete. $\square$

Apparently we can rewrite $\Phi(t, s) := \gamma(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i(t), t \in [0, \ell], s \in \mathbb{R}^{n-1}$ as

$$\Phi(t, S, s) = \gamma(t) + S\Big(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)\Big), t \in [0, \ell], s \in \mathbb{S}^{n-2}, S \geq 0.$$

We then rewrite the representation of Leading front in (4.2) in an equivalent way:

$$F_\gamma(t) = \{\Phi(t, S, s), S \geq 0, s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}\}. \tag{4.6}$$

For each $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$, define the scalar function,

$$S^\gamma_s(t) = \sup\{S \geq 0 : \Phi(t, S, s) \in \Omega\}. \tag{4.7}$$

That is, $S^\gamma_s(t)$ is the distance from $\gamma(t)$ to $\partial\Omega$ in the direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$. From the definition of $\Sigma^\gamma(t)$ and Σ^γ,

$$\Sigma^\gamma(t) = \{(S, s) : s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S^\gamma_s(t)\}, \tag{4.8}$$

and

$$\Sigma^\gamma = \{(t, S, s), t \in [0,\ell], s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\}. \tag{4.9}$$

Since $|\gamma'(t)| = 1$, $\gamma''(t) \cdot \gamma'(t) = 0$, we can then write

$$\gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t) \mathbf{N}_i(t).$$

Similarly we can also write

$$\mathbf{N}_i' = \kappa_{i_0} \gamma' + \sum_{j=1}^{n-1} \kappa_{i_j} \mathbf{N}_j.$$

It is easy to see that $\kappa_{i_0} = -\kappa_i$, $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. These equations can be written as the matrix equation

$$\begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} \\ -\kappa_1 & 0 & \kappa_{1_2} & \cdots & \kappa_{1_{n-1}} \\ -\kappa_2 & -\kappa_{1_2} & 0 & \cdots & \kappa_{2_{n-1}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix} \tag{4.10}$$

Given two *non-parallel* leading front $F_\gamma(t)$ and $F_\gamma(\tilde{t})$, denote their intersection-a $(n-2)$ dimensional plane $F(t, \tilde{t})$. Given $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$, define $L_s(t, \tilde{t})$ as the distance from $\gamma(t)$ to $F(t, \tilde{t})$ along the direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$ (we set $L_s(t, \tilde{t}) = +\infty$ if it does not hit $F(t, \tilde{t})$ along this direction) (Figure 15). We then define,

$$L_s^\gamma(t) := \inf\{L_s(t, \tilde{t}) : \tilde{t} \neq t\}. \tag{4.11}$$

Since all $F(t, \tilde{t})$ are outside Ω, $L_s^\gamma(t) \geq S_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$.

Lemma 4.7. $L_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i \kappa_i(t)\Big) \leq 1$ *for all* $t \in [0, \ell]$ *and* $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$.

Proof. Suppose $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel, we equate the representation of these two Leading fronts,

$$\gamma(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i(t) = \gamma(\tilde{t}) + \sum_{i=1}^{n-1} r_i \mathbf{N}_i(\tilde{t}).$$

This is a linear system of n equations and $2n-2$ unknowns $(s_1, \cdots, s_{n-1}, r_1, \cdots, r_{n-1})$. Solution for this system of equations exists because the two Leading front are not parallel. Then direct

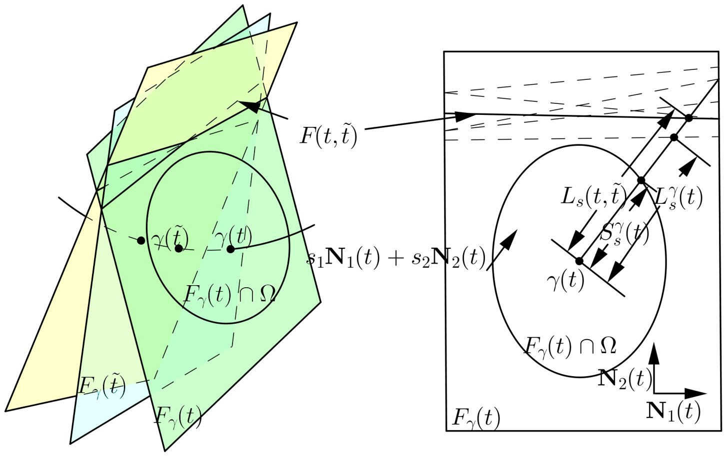

FIGURE 15.

computation using Cramer's rule gives the formula for $F(t,\tilde{t})$ explicitly,

$$F(t,\tilde{t}) = \{x \in F_\gamma(t) : (x-\gamma(t)) \cdot \Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \mathbf{N}_i(t) \Big) = 1\},$$

where

$$h_i(t,\tilde{t}) := \det[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \mathbf{N}_i(t)]$$

for $1 \le i \le n-1$, and

$$H(t,\tilde{t}) = \det[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \gamma(t)-\gamma(\tilde{t})]$$

Note that $H(t,\tilde{t}) \neq 0$ since $\gamma(t)-\gamma(\tilde{t})$ is not parallel to $F_\gamma(\tilde{t})$.

We first claim that

$$L_s(t,\tilde{t}) \Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i \Big) \le 1. \tag{4.12}$$

Indeed, we divide the situation into two cases. In the first case, suppose we travel from $\gamma(t)$ along a given direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$ and hit $F(t,\tilde{t})$, then for $x \in F(t,\tilde{t})$,

$$x - \gamma(t) = L_s(t,\tilde{t}) \Big(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t) \Big).$$

Therefore,

$$L_s(t,\tilde{t}) \Big(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t) \Big) \cdot \Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \mathbf{N}_i(t) \Big) = L_s(t,\tilde{t}) \Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i \Big) = 1. \tag{4.13}$$

Suppose for a certain direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$ we do not hit $F(t,\tilde{t})$, in which case we set $L_s(t,\tilde{t}) = +\infty$, then we must hit $F(t,\tilde{t})$ through the direction $-\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$, therefore, by (4.13),

$$L_{-s}(t,\tilde{t})\Big(\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) = 1.$$

In particular, since $L_{-s}(t,\tilde{t}) > 0$,

$$\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i > 0.$$

We then must have,

$$L_s(t,\tilde{t})\Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) < 0. \tag{4.14}$$

(4.13) and (4.14) together gives that in either case (4.12) holds true, which proves our claim.

We secondly claim,

$$L_s^\gamma(t)\Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) \le 1 \tag{4.15}$$

for all $t, \tilde{t} \in [0,\ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if for a given $t, \tilde{t}$ and $s \in \mathbb{S}^{n-2}$, $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel, and

$$-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i \ge 0,$$

then,

$$L_s^\gamma(t)\Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) \le L_s(t,\tilde{t})\Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) = 1$$

which gives (4.15) for this case. If for a certain $t, \tilde{t}$ and $s \in \mathbb{S}^{n-2}$, $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are still not parallel, but

$$-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i < 0,$$

then

$$L_s^\gamma(t)\Big(-\sum_{i=1}^{n-1} \frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} s_i\Big) < 0,$$

hence (4.15) is obviously satisfied. Finally, if $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are parallel, then $h_i(t,\tilde{t}) = 0$ for all $1 \le i \le n-1$, hence the (4.15) is again satisfied. Therefore (4.15) is true for all $t, \tilde{t} \in [0,\ell]$ and $s \in \mathbb{S}^{n-2}$.

We thirdly claim that

$$-\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \to \kappa_i(t), 1 \le i \le n-1. \tag{4.16}$$

as $\tilde{t} \to t$. Indeed, $H(t,\tilde{t}) \approx t - \tilde{t}$ as $\tilde{t} \to t$. Moreover,

$$h_i(t,t) = \det[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = 0.$$

Then,

$$\begin{aligned} -\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} &\approx -\frac{h_i(t,\tilde{t}) - h_i(t,t)}{t-\tilde{t}} \to \\ &\det[\mathbf{N}_1'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] + \cdots + \det[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}'(t), \mathbf{N}_i(t)]. \end{aligned} \tag{4.17}$$

Recall that

$$\mathbf{N}_i' = \kappa_{i_0}\gamma' + \sum_{j=1}^{n-1} \kappa_{i_j}\mathbf{N}_j$$

with $\kappa_{i_0} = -\kappa_i$, $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. Plug this expression into (4.17) and it is easy to see that all other terms vanish except

$$\det[\mathbf{N}_1(t), \cdots, \mathbf{N}_i'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = -\kappa_i \det[\mathbf{N}_1(t), \cdots, \gamma'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = \kappa_i$$

because $\det[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \gamma'(t)] = 1$. This proves (4.16).

Passing in (4.15) to the limit $\tilde{t} \to t$ we obtain the lemma. The proof is complete. □

Recall $S_s^\gamma(t)$ as defined in (4.7) satisfies $S_s^\gamma(t) \le L_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ due to the fact that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = \emptyset$ for all $t, \tilde{t} \in [0,\ell], \tilde{t} \ne t$, we then have,

Corollary 4.8. *$S_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \le 1$ for all $t \in [0,\ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$.*

Proof. If $\sum_{i=1}^{n-1} s_i\kappa_i(t) \ge 0$, then

$$S_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \le L_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \le 1. \tag{4.18}$$

If $\sum_{i=1}^{n-1} s_i\kappa_i(t) < 0$, then the result is obviously true. □

From the definition of Φ in (4.1), Φ is Lipschitz, hence its Jacobian $J_\Phi = \det D\Phi$ exists a.e. on Σ^γ, where Σ^γ has two equivalent representation (4.5) and (4.9). We will show the Corollary 4.8 implies $J_\Phi > 0$ a.e. on Σ^γ where

Lemma 4.9. *$J_\Phi(t,s) = 1 - \sum_{i=1}^{n-1} s_i\kappa_i(t) > 0$ for all $(t,s) \in \Sigma^\gamma$.*

Proof. Differentiating $\Phi(t,s)$ with respect to $(t, s_1, \cdots, s_{n-1})$ gives,

$$J_\Phi(t,s) = \det[\gamma'(t) + \sum_{i=1}^{n-1} s_i\mathbf{N}_i'(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t)]. \tag{4.19}$$

Substitute the expression of $\gamma'(t), \mathbf{N}_1'(t), \cdots, \mathbf{N}_{n-1}'(t)$ as linear combinations of $\gamma(t), \mathbf{N}_1'(t), \cdots, \mathbf{N}_{n-1}(t)$ into (4.19), we obtain, after Gaussian elimination, that,

$$J_\Phi(t,s) = 1 - \sum_{i=1}^{n-1} s_i\kappa_i(t). \tag{4.20}$$

If $\sum_{i=1}^{n-1} s_i\kappa_i(t) \le 0$, then obviously $J_\Phi(t,s) > 0$. Suppose now $\sum_{i=1}^{n-1} s_i\kappa_i(t) > 0$. By (4.9), $\Sigma^\gamma = \{(t, S, s), t \in [0,\ell], s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\}$, thus,

$$\sum_{i=1}^{n-1} s_i\kappa_i(t) = |s|\Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|}\kappa_i(t)\Big) < S_s^\gamma(t)\Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|}\kappa_i(t)\Big) \le 1$$

by Corollary 4.8. Therefore, $J_\Phi(t,s) > 0$ for all $(t,s) \in \Sigma^\gamma$. The proof is complete. □

4.3. **Moving Frames in the target space.** We are now in a position to define the moving frame in the target space $\mathbb{R}^{n+1}$. Let $\mathbf{N}_i(t), 1 \le i \le n-1$ be as in Subsection 4.2. Define the leading curve corresponding to γ in $u(\Omega(\gamma))$ to be

$$\tilde{\gamma} := u \circ \gamma.$$

Recall we defined in (4.1) that

$$\Phi(t,s) = \gamma(t) + \sum_{i=1}^{n-1} s_i\mathbf{N}_i(t),$$

and in (4.4) that

$$P_\gamma(t) = \{\Phi(t,s), s = (s_1, \cdots, s_{n-1}) \in \Sigma^\gamma(t)\}.$$

We also recall from Subsection 4.1 that ∇u is constant on $P_\gamma(t)$ for each $t \in [0,\ell]$. Hence for each $t \in [0,\ell]$, $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$.

Consider the Darboux frame $(\tilde{\gamma}', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$,

$$\begin{aligned} \tilde{\gamma}'(t) &= \tilde{\gamma}'(t) \\ \mathbf{v}_1(t) &= \nabla u(\gamma(t))\mathbf{N}_1(t) \\ &\cdots \\ \mathbf{v}_{n-1}(t) &= \nabla u(\gamma(t))\mathbf{N}_{n-1}(t) \\ \mathbf{n}(t) &= \tilde{\gamma}'(t) \times \mathbf{v}_1(t) \times \cdots \times \mathbf{v}_{n-1}(t). \end{aligned}$$

Since u is an isometric affine map along $P_\gamma(t)$ for each $t \in [0, \ell]$ we obtain

$$u(\Phi(t,s)) = \tilde{\gamma}(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i(t) \tag{4.21}$$

for all $t \in [0, \ell]$ and $s \in \Sigma^\gamma(t)$. Differentiating with respect to t we get

$$\nabla u(\Phi(t,s))\Big(\gamma'(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i'(t)\Big) = \tilde{\gamma}'(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i'(t), \tag{4.22}$$

and differentiating with respect to $(s_1, \cdots, s_{n-1})$ we obtain,

$$\nabla u(\Phi(t,s))\mathbf{N}_i(t) = \mathbf{v}_i(t). \tag{4.23}$$

Therefore by (4.10) we have

$$\mathbf{N}_i' = -\kappa_i \gamma' - \kappa_{1_i}\mathbf{N}_1 - \cdots + 0 \cdot \mathbf{N}_i + \kappa_{i_{i+1}}\mathbf{N}_{i+1} + \cdots + \kappa_{i_{n-1}}\mathbf{N}_{n-1},$$

and together with (4.22) and (4.23) we get

$$\begin{aligned} \tilde{\gamma}'(t) + s_1\mathbf{v}_1'(t) + \cdots + s_{n-1}\mathbf{v}_{n-1}'(t) &= \nabla u(\Phi(t,s))\Big(1 - \sum_{i=1}^{n-1} s_i\kappa_i(t)\Big)\gamma'(t) \\ &\quad + s_1(\kappa_{1_2}(t)\mathbf{v}_1(t) + \cdots + \kappa_{1_{n-1}}(t)\mathbf{v}_{n-1}(t)) \\ &\quad + \cdots \\ &\quad + s_{n-1}(-\kappa_{1_{n-1}}(t)\mathbf{v}_1(t) + \cdots + 0 \cdot \mathbf{v}_{n-1}(t)). \end{aligned} \tag{4.24}$$

Also, by (4.22), for $s = 0$ we have

$$\nabla u(\Phi(t,0))\gamma'(t) = \tilde{\gamma}'(t).$$

Since $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$ for each $t \in [0, \ell]$, we obtain

$$\nabla u(\Phi(t,s))\gamma'(t) = \nabla u(\Phi(t,0))\gamma'(t) = \tilde{\gamma}'(t). \tag{4.25}$$

Alongside (4.23), this shows that at each point in $\Omega(\gamma)$, ∇u maps an orthonormal frame to another orthonormal frame and this orthonormal frame only depends on t. Finally, matching coefficients in (4.24) yields

$$\mathbf{v}_i' = -\kappa_i \tilde{\gamma}' - \kappa_{1_i}\mathbf{v}_1 - \cdots + 0 \cdot \mathbf{v}_i + \kappa_{i_{i+1}}\mathbf{v}_{i+1} + \cdots + \kappa_{i_{n-1}}\mathbf{v}_{n-1}.$$

In other words, the following system of ODEs is satisfied by the Darboux frame of $\tilde{\gamma}$:

$$\begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} & \kappa_{\mathbf{n}} \\ -\kappa_1 & 0 & \kappa_{1_2} & \cdots & \kappa_{1_{n-1}} & 0 \\ -\kappa_2 & -\kappa_{1_2} & 0 & \cdots & \kappa_{2_{n-1}} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & \cdots & 0 & 0 \\ -\kappa_{\mathbf{n}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix} \tag{4.26}$$

4.4. **Change of variable formula.** Recall that $\Phi : \Sigma^\gamma \to \Omega(\gamma)$ is one-to-one and onto, where Σ^γ was defined in (4.5), For $(t,s) \in \Sigma^\gamma$, let $u_i(t,s) := (\frac{\partial}{\partial x_i} u) \circ \Phi(t,s)$, note that u_i is the ith column of $\nabla u \circ \Phi$. The following holds for all $(t,s) \in \Sigma^\gamma$ and for simplicity we omit it. Since $\nabla u^T \mathbf{n} \cdot \gamma' = \mathbf{n} \cdot \nabla u \gamma' = \mathbf{n} \cdot \tilde{\gamma}' = 0$ and $\nabla u^T \mathbf{n} \cdot \mathbf{N}_j = \mathbf{n} \cdot \nabla u \mathbf{N}_j = \mathbf{n} \cdot \mathbf{v}_j = 0$ for all $1 \le j \le n-1$, we have $\nabla u^T \mathbf{n} = 0$, i.e. $u_i \cdot \mathbf{n} = 0$ for all $1 \le i \le n$. Thus,

$$\begin{aligned} u_i = \big(u_i \cdot \tilde{\gamma}'\big)\tilde{\gamma}' + \sum_{j=1}^{n-1} \big(u_i \cdot \mathbf{v}_j\big)\mathbf{v}_j + \big(u_i \cdot \mathbf{n}\big)\mathbf{n} \quad &= (u_i \cdot \tilde{\gamma}')\tilde{\gamma}' + \sum_{j=1}^{n-1} (u_i \cdot \mathbf{v}_j)\mathbf{v}_j \\ &= (u_i \cdot \nabla u \gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1} (u_i \cdot \nabla u \mathbf{N}_j)\mathbf{v}_j \\ &= (\nabla u^T u_i \cdot \gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1} (\nabla u^T u_i \cdot \mathbf{N}_j)\mathbf{v}_j \\ &= (\mathbf{e}_i \cdot \gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1} (\mathbf{e}_i \cdot \mathbf{N}_j)\mathbf{v}_j, \end{aligned} \tag{4.27}$$

where $\mathbf{e}_i = (0, .., 1, \cdots, 0)$. Note that the right hand side of (4.27) is independent of s. Differentiating with respect to $s = (s_1, \cdots, s_{n-1})$ we get,

$$(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t,s))\mathbf{N}_i(t) = 0 \text{ for all } i. \tag{4.28}$$

Differentiating u_i with respect to t we obtain,

$$\begin{aligned} &(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t,s))(\gamma'(t) + \sum_{j=1}^{n-1} s_j \mathbf{N}_j'(t)) \\ &\quad = (\mathbf{e}_i \cdot \gamma''(t))\tilde{\gamma}'(t) + (\mathbf{e}_i \cdot \gamma'(t))\tilde{\gamma}''(t) + \sum_{j=1}^{n-1} (\mathbf{e}_i \cdot \mathbf{N}_j'(t))\mathbf{v}_j(t) + \sum_{j=1}^{n-1} (\mathbf{e}_i \cdot \mathbf{N}_j(t))\mathbf{v}_j'(t). \end{aligned} \tag{4.29}$$

If we write out $\mathbf{N}_i'$ as a linear combination of γ' and $\mathbf{N}_i, i = 1, \cdots n-1$, the left hand side of (4.29) becomes

$$(1 - \sum_{j=1}^{N-1} s_j \kappa_j(t))(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t,s))\gamma'(t).$$

If we write out $\tilde{\gamma}''$ and $\mathbf{v}_j'$, $j = 1, \cdots, n-1$ as a linear combination of $\tilde{\gamma}'$ and $\mathbf{v}_j$, $j = 1, \cdots, n-1$ and $\mathbf{n}$ as in (4.26), it is easy to see that all terms on the right hand side of (4.29) cancel each other except for $(\mathbf{e}_i \cdot \gamma'(t))\kappa_{\mathbf{n}}(t)\mathbf{n}(t)$. By Lemma 4.9, $1 - \sum_{j=1}^{n-1} s_j\kappa_j(t) > 0$ for all $(t, s) \in \Sigma^\gamma$. Therefore,

$$(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t,s))\gamma'(t) = \frac{(\mathbf{e}_i \cdot \gamma'(t))\kappa_{\mathbf{n}}(t)\mathbf{n}(t)}{1 - \sum_{j=1}^{n-1} s_j\kappa_j(t)}. \tag{4.30}$$

Since Φ is Lipschitz with $J_\Phi(t,s) = 1 - \sum_{j=1}^{n-1} s_j\kappa_j(t) > 0$, change of variable $x = \Phi(t,s)$ with (4.21) and (4.30) yield,

$$\int_{\Omega(\gamma)} |u(x)|^2 dx = \int_0^\ell \int_{\Sigma^\gamma(t)} |\tilde{\gamma}(t) + \sum_{i=1}^{n-1} s_i\mathbf{v}_i(t)|^2 \cdot \big(1 - \sum_{i=1}^{n-1} s_i\kappa_i(t)\big) d\mathcal{H}^{n-1}(s)dt. \tag{4.31}$$

$$\int_{\Omega(\gamma)} |\nabla u(x)|^2 dx = n|\Omega(\gamma)|. \tag{4.32}$$

$$\begin{aligned}\int_{\Omega(\gamma)} |\nabla^2 u(x)|^2 dx &= \int_0^\ell \int_{\Sigma^\gamma(t)} \frac{\sum_i (\mathbf{e}_i \cdot \gamma'(t))^2\kappa_{\mathbf{n}}^2(t)}{\big(1 - \sum_{i=1}^{n-1} s_i\kappa_i(t)\big)} d\mathcal{H}^{n-1}(s)dt \\ &= \int_0^\ell \int_{\Sigma^\gamma(t)} \frac{\kappa_{\mathbf{n}}^2(t)}{\big(1 - \sum_{i=1}^{n-1} s_i\kappa_i(t)\big)} d\mathcal{H}^{n-1}(s)dt.\end{aligned} \tag{4.33}$$

4.5. **Approximation process for $u|_{\Omega(\gamma)}$.** Recall from (4.11) that for a given $t \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$,

$$L_s^\gamma(t) := \inf\{L_s(t, \tilde{t}) : \tilde{t} \neq t\}, \tag{4.34}$$

where $L_s(t, \tilde{t})$ is the distance from $\gamma(t)$ to the intersection of two leading fronts $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ along direction $\sum_{i=1}^{n-1} s_i\mathbf{N}_i(t)$. Also recall $S_s^\gamma(t)$ defined in (4.7) is the distance from $\gamma(t)$ to $\partial\Omega$ in the same direction. Since all Leading fronts meet outside Ω, $L_s^\gamma(t) \geq S_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$.

Lemma 4.10. *There exists a sequence of isometries $u_m \in W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ converging strongly to u with the property that each u_m has a suitable leading curve $\gamma_m : [0, \ell_m] \to \mathbb{R}^n$ for which $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell_m]$.*

Proof. The proof is exactly the same as the 2-dimensional case, [35], proposition 3.2, because its proof is independent of dimensions. For the sake of completeness we include it here.

Consider $D_m : \mathbb{R}^n \to \mathbb{R}^n$ as the dilation centered at $x_0 = \gamma(0)$ by

$$D_m(x) := \frac{m}{m-1}(x - x_0) + x_0.$$

and as a correspondence, $\tilde{D}_m : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as the dilation centered at $y_0 = u(x_0)$ by

$$\tilde{D}_m(y) := \frac{m}{m-1}(y - y_0) + y_0.$$

Let $\Omega_m(\gamma) = D_m(\Omega(\gamma))$ and $\tilde{u}_m : \Omega_m(\gamma) \to \mathbb{R}^{n+1}$ as

$$\tilde{u}_m := \tilde{D}_m \circ u \circ D_m^{-1}$$

Notice that $\Omega(\gamma) \subset \Omega_m(\gamma)$ (figure 15), so $\tilde{u}_m$ is well defined over $\Omega(\gamma)$, the sequence $\tilde{u}_m \to u$ in $W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ and it is an isometric immersion. However, we still need some further construction to have a suitable leading curve γ_m that satisfies $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$. The curve

$$\gamma_m(t) := D_m \circ \gamma(\frac{m-1}{m}t).$$

defined on $[0, \frac{m}{m-1}\ell]$ is a leading curve for $\tilde{u}_m$, put,

$$\ell_m^* := \sup\{t : \gamma_m(t) \in \Omega(\gamma) \text{ and } F_{\gamma_m}(t) \cap F_\gamma(\ell) \cap \overline{\Omega} = \emptyset\}.$$

Finally we define our desired sequence of isometric immersion u_m as $\tilde{u}_m$ for the region of $\Omega(\gamma)$ covered by $F_{\gamma_m}(t), 0 \le t \le \ell_m^* - 1/m$ and extend by affine extension to the entire $\Omega(\gamma)$ (Figure 16), i.e.,

$$u_m(x) = \begin{cases} \tilde{u}_m(x) \quad \text{if } x \in F_{\gamma_m}(t) \text{ for } 0 \le t \le \ell_m^* - \frac{1}{m} \\ \nabla \tilde{u}_m\Big(\gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big)\Big(x - \gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big) \\ +\tilde{u}_m\Big(\gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big) \quad \text{otherwise.} \end{cases} \tag{4.35}$$

It is obvious each u_m admits a leading curve (still denoted by γ_m) satisfying $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell_m]$. The proof is complete. □

Remark 4.11. By the above Lemma, we can just assume u has a suitable Leading curve γ that satisfies $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$.

Lemma 4.12. *Suppose $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$. Then there is a sequence of smooth maps in $I^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ converging strongly to u.*

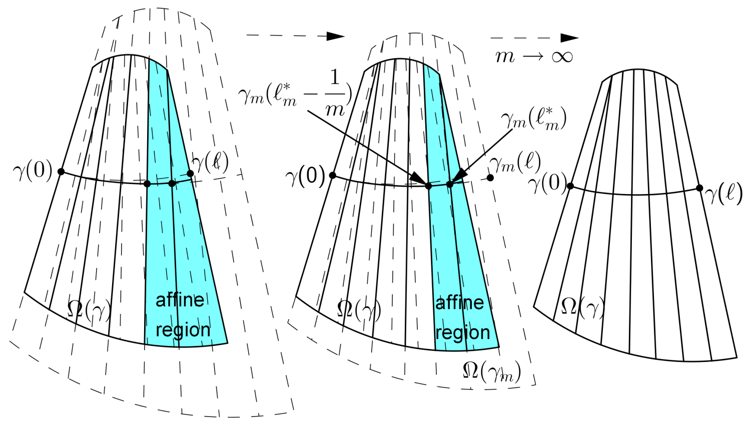

FIGURE 16.

Proof. The idea of construction is to construct a smooth curve γ_m approximating γ. We do not know yet this curve is a Leading curve of u_m or not, so we cannot call the $(n-2)$-dimensional hyperplane orthogonal to γ_m at t Leading fronts. Instead we call them *orthogonal fronts* and denote them by $F_{\gamma_m}(t)$. If we manage to show all such orthogonal fronts meet outside $\Omega(\gamma_m)$, γ_m becomes a Leading curve for u_m and $F_{\gamma_m}(t)$ are actually the Leading fronts. We then define u_m to be isometric affine mapping along each Leading front $F_{\gamma_m}(t)$. Since all the Leading fronts intersect outside Ω, u_m is well-defined.

We first need the following lemma,

Lemma 4.13. *There exists smooth curve γ_m such that $\gamma_m(t) \to \gamma(t)$ in $W^{2,p}([0,\ell], \mathbb{R}^n)$ for all $1 \le p < \infty$ and satisfies $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$.*

Proof. The construction is technical. For the convenience of the readers we present the main steps here and present a detailed proof in Appendix A. The proof needs six steps:

Step 1. There exists a smooth curve $\Gamma_m : [0,\ell] \to \mathbb{R}^n$ and an orthonormal frame $(\Gamma'_m(t), \mathbf{N}_{1,m}(t), \cdots, \mathbf{N}_{n-1,m}(t))$ with initial condition $\gamma_m(0) = \gamma(0)$, $\Gamma'_m(0) = \gamma'(0)$, and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$ such that $\Gamma_m \to \gamma$ in $W^{2,p}([0,\ell], \mathbb{R}^n)$ for all $1 \le p \le \infty$ and $(\Gamma'_m, \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots, \mathbf{N}_{n-1})$ uniformly. To obtain this, we simply approximate $\kappa_i, \kappa_{i_j}, 1 \le i,j \le n-1$ by uniformly bounded smooth functions $\tilde{\kappa}_{i,m}, \kappa_{i_j,m}, 1 \le i,j \le n-1$ and solve the ordinary differential equations in (4.10) with respect to the matrix whose entries are these smooth functions. Uniform convergence of the moving frame is due to a result by Opial [34] for L^1 convergence parameters.

However Γ_m is not our desired curve since we cannot guarantee all its orthogonal fronts intersect outside $\overline{\Omega(\gamma)}$. This happens if Γ_m is too "curvy". We need to "flatten" its curvature continuously.

Step 2. We construct $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ continuous on $t \in [0, \ell]$ such that for each $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t)) \le 1,$$

where as before,

$$S_s^{\Gamma_m}(t) = \sup\{S \ge 0 : \Gamma_m(t) + S\Big(\sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t)\Big) \in \Omega\}.$$

To obtain this, we simply cut-off the curvature. That is, we define,

$$\lambda_m(t) := \min\Big\{1, 1/(\sup_{|s|=1}\{(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))\})\Big\}$$

where $\kappa_{i,m}, 1 \le i \le n-1$ are those found in Step 1. A first observation is that $0 < \lambda_m \le 1$. Indeed, there must exist $s \in \mathbb{S}^{n-2}$ such that $\displaystyle\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \ge 0$, so the supreme over all $s \in \mathbb{S}^{n-2}$ must be nonnegative. On the other hand, $S_s^{\Gamma_m}$ as well as all $\tilde{\kappa}_{i,m}$ are bounded so λ_m is bounded below by a positive number.

A second observation is λ_m is continuous, this follows from the fact that

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))$$

is uniformly continuous on $(s,t) \in \mathbb{S}^{n-2} \times [0, \ell]$ by the implicit function theorem for C^1 functions, since $S_s^{\Gamma_m}(t)$ depends continuously on the intersection of $F_{\Gamma_m}(t)$ with $\partial\Omega$ and Ω is a C^1 domain. A simply argument then shows $h(t) := \displaystyle\sup_{|s|=1}\{(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))\}$ is continuous.

We then define a vector valued function $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ as

$$(\tilde{\tilde{\kappa}}_{1,m}(t), \cdots, \tilde{\tilde{\kappa}}_{n-1,m}(t)) := \lambda_m(t)(\tilde{\kappa}_{1,m}(t), \cdots, \tilde{\kappa}_{n-1,m}(t))$$

It follows from the definition of λ_m that

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t)) \le 1.$$

Step 3. We show the a.e. convergence of the continuous vector valued function $\tilde{\tilde{\kappa}}_m \to \kappa = (\kappa_1, \cdots \kappa_{n-1})$. To show this, we need to show $\lambda_{m_{k(m)}}$ defined in step 2 goes to 1 a.e. The difficult

part is to prove

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } s \in \mathbb{S}^{n-2} \text{ and } t \in [0,\ell].$$

This result is also due to the implicit function theorem for C^1 functions.

Step 4. Due to the fact that $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ are continuous, for each m we can find smooth vector valued function κ_m such that $\kappa_m - \tilde{\tilde{\kappa}}_m \to 0$ uniformly for all $t \in [0,\ell]$, hence,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{4})(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1.$$

for m sufficiently large.

Step 5. We construct our desired curve γ_m as well as its moving frame $(\gamma_m', \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m})$ by solving the differential equation (4.10) with respect to $\kappa_m = (\kappa_{1,m}, \cdots, \kappa_{n-1,m})$ found in step 4 and $\kappa_{i_j,m}, 1 \leq i,j \leq n-1$ found in step 1 satisfying initial condition $\gamma_m'(0) = \gamma'(0)$ and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. Poincaré inequality for intervals show that $\gamma_m \to \gamma$ in $W^{2,p}([0,1], \mathbb{R}^n)$ for all $1 \leq p < \infty$ and $(\gamma_m', \mathbf{N}_{1,m}, \cdots \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots \mathbf{N}_{n-1})$ uniformly by Opial [34], Theorem 1. Moreover, by Step 4, κ_m satisfies

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{4})(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1.$$

However, we need to replace $S_s^{\Gamma_m}$ by $S_s^{\gamma_m}$. This is not a problem since by the implicit function theorem as in step 3, both $S_s^{\gamma_m}$ and $S_s^{\Gamma_m}$ converge to S_s^{γ} uniformly, thus for m sufficiently large,

$$(S_s^{\gamma_m}(t) + \frac{\rho}{8})(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1. \tag{4.36}$$

Step 6. We show all orthogonal fronts satisfy $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$.

For γ_m and its moving frame $(\gamma_m', \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m})$ found in Step 5, let $\Phi_m : [0,\ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi_m(t,s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t).$$

Let $\Sigma^{\gamma_m} = \{(t,s) : \Phi_m(t,s) \in \overline{\Omega}\}$. By the same argument as Lemma 4.6, Φ_m maps Σ^{γ_m} onto $\overline{\Omega(\gamma_m)}$ where $\Omega(\gamma_m)$ is the subset of Ω covered by all orthogonal fronts $F_{\gamma_m}(t), t \in [0,\ell]$.

Let $d := \operatorname{diam}(\Omega)$, it can be shown that (4.36) implies

$$J_{\Phi_m}(t,s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \geq \min\{\rho/16d, 1/2\} \tag{4.37}$$

for all $(t,s) \in \Sigma^{\gamma_m}$. By Inverse function theorem due to Clarke [5], Φ admits a local Lipschitz inverse, actually a *global* Lipschitz inverse since the Jacobian is everywhere bounded below by a

positive constant in Σ^{γ_m}. In particular, Φ_m is one-to-one on Σ^{γ_m}. This implies all orthogonal front $F_{\gamma_m}(t), t \in [0,\ell]$ meets outside $\overline{\Omega}$. The proof of Lemma 4.13 is complete. $\square$

We also need to define the curves $\tilde{\gamma}_m$ in the target space $u(\Omega(\gamma))$ corresponding to γ_m. Recall that the normal curvature $\kappa_{\mathbf{n}}$ defined in (4.26) is bounded. We choose a sequence of uniformly bounded smooth function $\tilde{\kappa}_{\mathbf{n},m}$ such that $\tilde{\kappa}_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$ a.e. in $[0,\ell]$, (and hence in L^p for all $1 \le p < \infty$).

We need to flatten $\tilde{\kappa}_{\mathbf{n},m}$ around the end points 0 and ℓ for two reasons: first, it might happen that $\Omega(\gamma) \nsubseteq \Omega(\gamma_m)$ so we need to extend the isometric immersion defined on $\Omega(\gamma_m)$ smoothly to the region of $\Omega(\gamma)$ outside $\Omega(\gamma_m)$. Second, so far all the construction is on one covered domain $\Omega(\gamma)$ and our final goal is to glue all the different covered domains together smoothly. By flattening $\tilde{\kappa}_{\mathbf{n},m}$ around the end point 0 and ℓ, u_m constructed later is affine near the Leading planes $P_\gamma(0)$ and $P_\gamma(\ell)$ (for definition of leading planes see Definition 4.5) so that we can join all the pieces smoothly. The modification goes as follows: by (4.30), the second derivative of u vanishes whenever $\kappa_{\mathbf{n}} = 0$. Put

$$\ell_m^* = \begin{cases} \ell \quad \text{if} \quad \Omega(\gamma) \subset \Omega(\gamma_m) \text{ and,} \\ \sup\{t \in [0,\ell], F_{\gamma_m}(t) \cap F_\gamma(\ell) \cap \overline{\Omega}(\gamma) = \emptyset\} \text{ otherwise.} \end{cases}$$

By step 1 of Lemma 4.13, $F_{\gamma_m}(t) \to F_\gamma(t)$ uniformly, hence $\ell_m^* \to \ell$ as $m \to \infty$.

Let ψ_1 be any smooth positive function which is 0 on $[-1,\infty)$ and 1 on $(-\infty,-2)$. Let ψ_2 be any smooth positive function which is 0 on $(-\infty,1]$ and 1 on $(2,\infty)$. We put,

$$\kappa_{\mathbf{n},m}(t) := \psi_1(m(t-\ell_m^*))\psi_2(mt)\tilde{\kappa}_{\mathbf{n},m}(t), t \in [0,\ell]$$

and we solve the following linear system for initial values $\tilde{\gamma}'_m(0) = \tilde{\gamma}'(0)$, $\mathbf{v}_{i,m}(0) = \mathbf{v}_i(0)$, and $\mathbf{n}_m(0) = \mathbf{n}(0)$:

$$\begin{pmatrix} \tilde{\gamma}'_m \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} & \kappa_{\mathbf{n},m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} & 0 \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 & 0 \\ -\kappa_{\mathbf{n},m} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\gamma}'_m \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}$$

We define

$$\tilde{\gamma}_m(t) = \tilde{\gamma}(0) + \int_0^t \tilde{\gamma}'_m(\tau)d\tau.$$

By the same argument as in step 1 in the proof of Lemma 4.13, $\tilde{\gamma}_m \to \tilde{\gamma}$ in $W^{2,p}([0,\ell],\mathbb{R}^{n+1})$ and the moving frame $(\tilde{\gamma}'_m, \mathbf{v}_{1,m}, \cdots, \mathbf{v}_{n-1,m}, \mathbf{n}_m) \to (\tilde{\gamma}', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$ uniformly.

Eventually, we define our approximating sequence u_m on $\Omega(\gamma_m)$:

$$u_m(\gamma_m(t)+\sum_{i=1}^{n-1}s_i\mathbf{N}_{i,m}(t))=\tilde{\gamma}_m(t)+\sum_{i=1}^{n-1}s_i\mathbf{v}_{i,m}(t), \tag{4.38}$$

where γ_m is defined in Lemma 4.13. Such γ_m assures that all its leading fronts intersect outside $\overline{\Omega}$, hence u_m is well-defined and smooth over $\Omega(\gamma)\cap\Omega(\gamma_m)$.

As before, let $\Phi_m:[0,\ell]\times\mathbb{R}^{n-1}\to\mathbb{R}^n$ be defined as

$$\Phi_m(t,s)=\gamma_m(t)+\sum_{i=1}^{n-1}s_i\mathbf{N}_{i,m}(t),$$

and let $\Delta^{\gamma_m}=\{(t,s):\Phi_m(t,s)\in\Omega(\gamma)\}$. Same argument as Step 6 in Lemma 4.13 gives $\Phi_m(t,s)$ is a bi-Lipschitz mapping of Δ^{γ_m} onto $\Omega(\gamma)\cap\Omega(\gamma_m)$. By differentiating with respect to $t,s_1,\cdots,s_{n-1}$, same as (4.25) and (4.23), we see that at each point of x, $\nabla u_m(x)$ maps an orthonormal frame to an orthonormal frame. Hence $\nabla u_m(x)\in O(\mathbb{R}^n,\mathbb{R}^{n+1})$. Moreover, u_m is affine near $P_{\gamma_m}(\ell)$ and can be extended by an affine isometry over $\Omega(\gamma)$. Therefore, $u_m\in I^{2,2}(\Omega(\gamma),\mathbb{R}^n)$. Everything we have proved for isometric immersions of course applies, in particular, by (4.27), (4.30) (4.28) we have,

$$\frac{\partial}{\partial x_i}u_m\circ\Phi_m(t,s)=(\mathbf{e}_i\cdot\gamma_m'(t))\tilde{\gamma}_m'(t)+\sum_{j=1}^{n-1}(\mathbf{e}_i\cdot\mathbf{N}_{j,m}(t))\mathbf{v}_{j,m}(t), \tag{4.39}$$

$$(\nabla\frac{\partial}{\partial x_i}u_m)(\Phi_m(t,s))\gamma_m'(t)=\frac{(\mathbf{e}_i\cdot\gamma_m'(t))\kappa_{\mathbf{n},m}(t)\mathbf{n}(t)}{1-\sum_{i=1}^{n-1}s_i\kappa_{i,m}(t)},\ \text{and} \tag{4.40}$$

$$(\nabla\frac{\partial}{\partial x_i}u_m)(\Phi_m(t,s))\mathbf{N}_{i,m}(t)=0,\quad 1\le i\le n-1. \tag{4.41}$$

for all $t\in[0,\ell]$ and $s=(s_1,\cdots,s_{n-1})\in\Delta^{\gamma_m}(t)$.

Moreover, by (4.31),

$$\begin{aligned}\int_{\Omega(\gamma)}|u_m(x)|^2dx &= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)}|u_m(x)|^2dx+\int_{\Omega(\gamma)\setminus\Omega(\gamma_m)}|u_m(x)|^2dx\\ &= \int_0^\ell\int_{\Delta^{\gamma_m}(t)}|\tilde{\gamma}_m(t)+\sum_{i=1}^{n-1}s_i\mathbf{v}_{i,m}(t)|^2\cdot\Big(1-\sum_{i=1}^{n-1}s_i\kappa_{i,m}(t)\Big)d\mathcal{H}^{n-1}(s)dt\\ &\quad+\int_{\Omega(\gamma)\setminus\Omega(\gamma_m)}|u_m(\ell)+\nabla u_m(\ell)(x-\gamma_m(\ell)|^2dx.\end{aligned} \tag{4.42}$$

$$\int_{\Omega(\gamma)}|\nabla u_m(x)|^2dx=n|\Omega(\gamma)|. \tag{4.43}$$

$$
\begin{aligned}
\textstyle\int_{\Omega(\gamma)} |\nabla^2 u_m(x)|^2 dx &= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)} |\nabla^2 u_m(x)|^2 dx + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} |\nabla^2 u_m(x)|^2 dx \\
&= \int_0^\ell \int_{\Delta^{\gamma_m}(t)} \frac{\kappa^2_{\mathbf{n},m}(t)}{\left(1-\sum_{i=1}^{n-1} s_i\kappa_{i,m}(t)\right)} d\mathcal{H}^{n-1}(s)dt + 0.
\end{aligned}
\tag{4.44}
$$

It is easy to see $u_m \to u$ in $W^{2,2}(\Omega(\gamma),\mathbb{R}^{n+1})$ because $(\gamma_m', \mathbf{N}_{1,m}, \cdots \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots \mathbf{N}_{n-1})$ uniformly, $(\tilde\gamma_m', \mathbf{v}_{1,m}, \cdots, \mathbf{v}_{n-1,m}, \mathbf{n}_m) \to (\tilde\gamma', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$ uniformly, $\kappa_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$, $\kappa_{i,m} \to \kappa_i, 1 \le i \le n-1$ in $L^p([0,\ell])$ for all $1 \le p < \infty$, $1 - \sum_{i=1}^{n-1} s_i\kappa_{i,m}(t) \ge \min\{\rho/16d, 1/2\}$, $\Delta^{\gamma_m}(t) \to \Sigma^\gamma(t)$ for all $t \in [0,\ell]$ and $|\Omega(\gamma)\setminus\Omega(\gamma_m)| \to 0$. The proof is complete. □

Combining Lemmas 4.10 and 4.12 we get a smooth approximation sequence for any isometry u in $\Omega(\gamma)$.

4.6. **Approximation for u in Ω.** The proof is exactly the same as the proof in section 3.3 in [35]. For the sake of completeness we include the proof here.

Recall that we defined a maximal region on which u is affine a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω (recall Definition 3.9 for the definition of $(n-1)$-planes in Ω) and we have shown that we can assume Ω has only a finite number of bodies and is partitioned into bodies and covered domains. We call the maximal subdomain covered by some Leading curve γ an *arm*. Similar to Lemma 4.1 we also have,

Lemma 4.14. *It is sufficient to prove Theorem 1.5 for a function in $I^{2,2}(\Omega,\mathbb{R}^{n+1})$ with finite number of arms*

Proof. Since we have a finite number of bodies, the complement of bodies in $\tilde\Omega$ is a finite union of connected components $\cup_{j=1}^N \Delta_j$. Suppose one such region Δ is between two bodies B_1 and B_2, we want to show $\overline{\Delta}$ can be covered by a finite number of Leading curves.

Let us recall the definition of Leading planes (Definition 4.5). From our definition, each Leading plane is an open set with respect to the Leading front it belongs. Here we slight change the definition and still denote a Leading plane as its closure with respect to the Leading front it belongs. Since each $x \in \overline{\Delta}$ is covered by some leading curve, with our new definition of Leading planes, $\overline{\Delta}$ is a union of Leading planes by obvious modification of Lemma 4.6. For each Leading plane P, let $B^{n-1}(x^P, r^P)$ be the largest $n-1$ dimensional ball contain in P and we denote x^P the *center* of P. Since Δ is between two bodies and Δ is a convex domain,

$$r := \inf_P r^P > 0.$$

Let $\mathbf{N}$ be the normal vector field orthogonal to these Leading planes everywhere. Since none of the Leading planes intersect inside Δ by the definition of Leading curve, which has a Lipschitz boundary, the normal vector field approach each other in an Lipschitz angle. Therefore, we can choose an orientation of such that $\mathbf{N}$ is a Lipschitz vector fields.

Note that $\overline{\Delta} \cap B_1$ is a Leading plane and we denote it P_0. Let x_0 be the center of P_0 and let $\gamma_1 : [0, \ell_1] \to \overline{\Delta}$ be the unique maximal solution to the ODE,

$$\gamma_1'(t) = \mathbf{N}(\gamma_1(t)) \quad \gamma_1(0) = x_0. \tag{4.45}$$

If $\gamma_1(\ell_1) \in \Delta$, we can always find a unique leading curve $\gamma : [-\delta, \delta] \to \Delta$ with $\gamma(0) = \gamma_1(\ell_1)$ for some $\delta > 0$. Therefore, we can always prolong γ_1 inside Δ as long as it does not touch $\partial\Delta$, contraction to γ_1 being the maximal solution. Therefore, $\gamma_1(\ell_1) \in \partial\Delta$. Note that $\partial\Delta$ consists of components of ∂B_1, ∂B_2, and $\partial\Omega$. If $\gamma_1(\ell_1) \in \partial B_2$, then the entire Δ is covered by γ_1 and we are done. The only situation a different Leading curve is needed is when $\gamma_1(\ell_1) \in \partial\Omega \setminus \partial B_2$ (Figure 17).

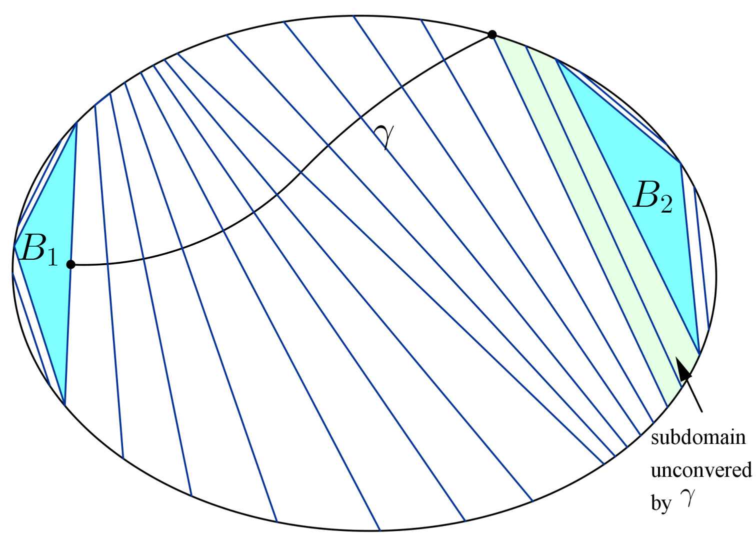

FIGURE 17.

In the latter case we consider the Leading plane P_1 in Ω passing through the point $\gamma_1(\ell_1)$. Such Leading plane P_1 is uniquely defined by Lemma 4.10. Let x_1 be the center of P_1 and let $\gamma_2 : [0, \ell_2]$ be the unique maximal solution to the ODE in (4.45) with initial condition $\gamma_2(0) = x_1$ and then we repeat the same argument as above.

We claim that a finite number of such $\gamma_i : [0, \ell_i]$ cover Δ. Indeed, denote the length of the curve $\gamma_i([0, \ell_i])$ by $|\gamma_i([0, \ell_i])|$, then as γ_i is parametrized by arc-length, $|\gamma_i([0, \ell_i])| = \ell_i$. Since

$\gamma_i(0) = x_{i-1}$, which is the center of P_{i-1}, and $\gamma_i(\ell_i) \in \partial\Omega$, the distance $|\gamma(\ell_i) - \gamma(0)| \geq \min\{r, \mathrm{dist}(x_0, \partial\Omega)\} =: c > 0$. Altogether we have,

$$\ell_i = |\gamma_i([0, \ell_i])| \geq |\gamma(\ell_i) - \gamma(0)| \geq c.$$

Now by the change of variable formula (4.31) and Remark 4.11 that $1 - \sum_{j=1}^{n-1} s_j \kappa_j(t) > \rho > 0$,

$$\Omega(\gamma_i) = \int_0^{\ell_i} \int_{\Sigma^{\gamma_i}(t)} 1 - \sum_{j=1}^{n-1} s_j \kappa_j(t) d\mathcal{H}^{n-1}(s) dt \geq C(n) c r^{n-1} \rho.$$

Hence i must be a finite number since Δ is bounded.

Two minor cases remain to be investigated. If Δ has common boundary with only one body, then we claim that there exists a sequence of isometric immersions $u_m \to u$ in $W^{2,2}(\Omega, \mathbb{R}^{n+1})$ and each u_m admits a finite number of arms that covers Δ. To do this we simply cut off the Leading planes corresponding to $\gamma_1(t), 0 \leq t \leq 1/m$ and extend the maps by an affine map to the whole region $\Omega(\gamma_1)$ as in (4.35). The idea is clear from Figure 18. If Δ does not have common boundary

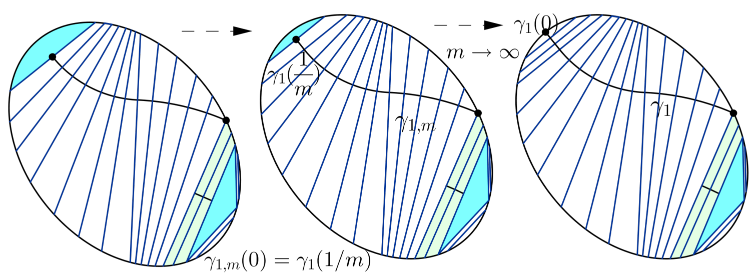

FIGURE 18. Approximation sequence with a finite number of arms.

with any bodies, then we divide Δ into two regions Δ_1 and Δ_2 by one Leading plane and apply the cut off argument to Δ_1 and Δ_2 individually. The proof is complete. □

Now since Ω is convex and simply-connected, we claim that two bodies are connected through one chain of bodies and arms: It suffices to consider the graph obtained by retracting bodies to vertices and arms to edges. This graph is simply connected because it is a deformation retract of Ω. Therefore every two vertices are connected through only one chain of edges, which proves the claim (Figure 19).

We begin by a central body B_1 and define our approximation sequence on each arm as in Subsection 4.5. Note that for this final purpose, we have constructed our approximation smooth isometric immersion to be affine near both ends, this allows us to apply an affine transformation to the target space of each arm so that the affine regions near its ends join together smoothly

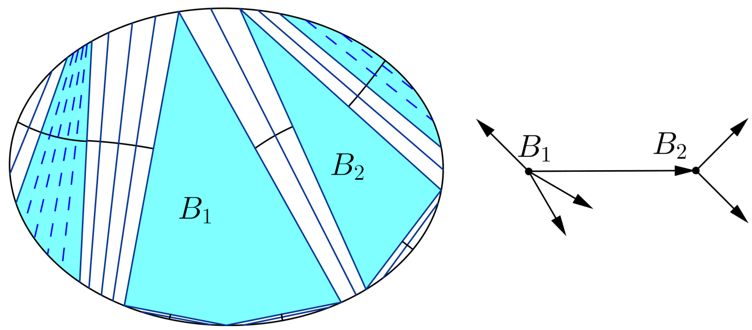

FIGURE 19. Graph of retraction of Ω.

all the way till we reach B_2. Meanwhile, we also apply an affine transformation to $u(B_2)$ so that it joins the last arm smoothly. It is easy to see from the uniform convergence of each term in representation (4.39) that such affine transformation converges to identity as $m \to 0$. Now we continue our construction using B_2 as a new starting point. Note that we will never come back to B_1 because they are connected through only one chain of arms. The construction of the approximating sequence on the entire domain Ω is complete. $\square$

APPENDIX A. DETAILED PROOF OF LEMMA 4.13

Step 1. Recall from the matrix of moving frame defined in Subsection 4.2. that $\gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t)\mathbf{N}_i(t))$, with κ_i bounded. We can choose uniformly bounded smooth functions $\tilde{\kappa}_{i,m} \to \kappa_i$ a.e. on $[0,\ell]$, and hence in measure due to the fact that $[0,\ell]$ is bounded. Since the sequence $\tilde{\kappa}_{i,m}$ are uniformly bounded, it follows $\tilde{\kappa}_{i,m} \to \kappa_i$ in L^p for all $1 \le p < \infty$. Similarly we can find uniformly bounded smooth functions $\kappa_{i_j,m} \to \kappa_{i_j}$ a.e. on $[0,\ell]$ (hence in L^p for all $1 \le p < \infty$) for $\kappa_{i_j}, 1 \le i,j \le n-1$. By solving the following system of ODEs:

$$\begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \begin{pmatrix} 0 & \tilde{\kappa}_{1,m} & \tilde{\kappa}_{2,m} & \cdots & \tilde{\kappa}_{n-1,m} \\ -\tilde{\kappa}_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} \\ -\tilde{\kappa}_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\tilde{\kappa}_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}$$

we obtain a unique orthogonal frame $(\Gamma'_m(t), \mathbf{N}_{1,m}(t), \cdots, \mathbf{N}_{n-1,m}(t))$ with initial condition $\Gamma'_m(0) = \gamma'(0)$, and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. We can then define

$$\Gamma_m(t) = \Gamma(0) + \int_0^t \Gamma'_m(\tau)d\tau.$$

We want to show $(\Gamma'_m, \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots, \mathbf{N}_{n-1})$ uniformly. This result is given by the following theorem due to Opial, [34], Theorem 1.

Lemma A.1 (Opial)**.** *Suppose the linear systems of differential equations,*

$$x'(t) = A_k(t)x(t), \quad x(0) = a_k, \quad k = 0, 1, 2, \cdots \tag{A.1}$$

admit a solution $x_k(t)$ in $[0, \ell]$ for all k. Suppose $a_k \to a_0$,

$$\int_0^t A_k(s)\,ds \to \int_0^t A_0(s)\,ds$$

uniformly for all $t \in [0, \ell]$ and A_k is a bounded sequence in L^1, i.e. $\sup_k \|A_k\|_{L^1([0,\ell])} < \infty$, then the solutions

$$x_k(t) \to x_0(t) \quad \textit{uniformly.}$$

Since $\tilde{\kappa}_{i,m} \to \kappa_i$ and $\kappa_{i_j,m} \to \kappa_{i_j}$ in L^p for all $1 \leq p < \infty$, in particular for $p = 1$, the conditions in Lemma A.1 are satisfied, hence $(\Gamma'_m, \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots, \mathbf{N}_{n-1})$ uniformly. Since $\Gamma''_m = \sum_{i=1}^{n-1} \tilde{\kappa}_{i,m}\mathbf{N}_{i,m}$, Γ''_m are uniformly bounded, and $\Gamma''_m \to \gamma''$ a.e. (and hence in L^p for all $1 \leq p < \infty$), Poincaré inequality for intervals implies that $\Gamma_m \to \gamma$ in $W^{2,p}[0, \ell], \mathbb{R}^n)$ for all $1 \leq p < \infty$.

However Γ_m is not our desired curve since we cannot guarantee that all its leading fronts intersect outside $\overline{\Omega}$. This happens if Γ_m is too "curvy". We need to "flatten" its curvature continuously. This needs to be done in several steps:

Step 2. We construct $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ continuous on $t \in [0, \ell]$ and for each $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t)) \leq 1 \tag{A.2}$$

where

$$S_s^{\Gamma_m}(t) = \sup\{S \geq 0 : \Gamma_m(t) + S\big(\sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t)\big) \in \Omega\}.$$

We first need the following lemma using implicit function theorem for C^1 functions.

Lemma A.2. *$S_s^{\Gamma_m}(t)$ is uniformly continuous on $(s, t) \in \mathbb{S}^{n-2} \times [0, \ell]$ and $S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t)$ uniformly on $(s, t) \in \mathbb{S}^{n-2} \times [0, \ell]$.*

Proof. Let $t_0 \in [0,\ell]$ and $s^0 = (s^0_1, \cdots, s^0_{n-1}) \in \mathbb{S}^{n-1}$ be arbitrary. We parametrize locally $\mathbb{S}^{n-2}$ by the polar coordinates: $s_i = s_i(\theta)$ where $\theta = (\theta_1, \cdots, \theta_{n-2}) \in U_1 \subset [0,\pi)^{n-3} \times [0, 2\pi)$. Let $\theta^0 \in U_1$ be such that $s^0_i = s_i(\theta^0)$.

Let $\gamma^0 = \gamma(t_0)$ and $\mathbf{N}^0_i = \mathbf{N}_i(t_0)$. Let x_0 be the intersection of the line segment $L = \{\gamma^0 + S\big(\sum_{i=1}^{n-1} s^0_i \mathbf{N}^0_i\big), 0 \le S\}$ and $\partial\Omega$. Then $x_0 = \gamma^0 + S_0\big(\sum_{i=1}^{n-1} s^0_i \mathbf{N}^0_i\big)$ for some $S_0 > 0$.

Since Ω is a C^1 domain, there exits an open subset of $U_2 \subset \mathbb{R}^{n-1}$ and a C^1 function $\alpha : U_2 \to \partial\Omega$ and $\alpha(\eta^0_1, \cdots, \eta^0_{n-1}) = x_0$ for some $(\eta^0_1, \cdots, \eta^0_{n-1}) \in U_2$.

Consider $F : \mathbb{R}^n \times \mathbb{R}^{n\times n-1} \times U_1 \times \mathbb{R} \times U_2 \to \mathbb{R}^n$

$$F(\gamma, \mathbf{N}_1, \cdots \mathbf{N}_{n-1}, \theta, S, \eta_1, \cdots, \eta_{n-1}) = \gamma + S\big(\sum_{i=1}^{n-1} s_i(\theta)\mathbf{N}_i\big) - \alpha(\eta_1, \cdots, \eta_{n-1}).$$

Since $x_0 \in \partial\Omega \cap L$,

$$F(\gamma^0, \mathbf{N}^0_1, \cdots, \mathbf{N}^0_{n-1}, \theta^0, S^0, \eta^0_1, \cdots, \eta^0_{n-1}) = 0.$$

Let

$$\mathbf{x} = (\gamma, \mathbf{N}_1, \cdots, \mathbf{N}_{n-1}, \theta), \quad \mathbf{y} = (S, \eta_1, \cdots, \eta_{n-1})$$

and

$$\alpha_k := \frac{\partial\alpha}{\partial\eta_k}, 1 \le k \le n-1.$$

Then

$$\det\big[(\frac{\partial F}{\partial \mathbf{y}}(\gamma^0, \mathbf{N}^0_1, \cdots, \mathbf{N}^0_{n-1}, \theta^0, S^0, \eta^0_1, \cdots, \eta^0_{n-1})\big] =$$

$$\det\big[\sum_{i=1}^{n-1} s_i(\theta^0)\mathbf{N}^0_i, \alpha_1(\eta^0_1, \cdots, \eta^0_{n-1}), \cdots, \alpha_{n-1}(\eta^0_1, \cdots, \eta^0_{n-1})\big] \neq 0.$$

Otherwise, the line segment L would be parallel to the tangent plane of $\partial\Omega$ at x_0, which is not possible since Ω is convex.

By implicit function theorem, there is an open neighborhood $V_1 \subset \mathbb{R}^n \times \mathbb{R}^{n\times n-1} \times U_1$ of $\mathbf{x}_0 = (\gamma^0, \mathbf{N}^0_1, \cdots, \mathbf{N}^0_{n-1}, \theta^0)$, $V_2 \subset \mathbb{R} \times U_2$ of $\mathbf{y}_0 = (S^0, \eta^0_1, \cdots, \eta^0_{n-1})$, and a C^1 diffeomorphism $\mathbf{y} : V_1 \to V_2$ such that

$$F(\mathbf{x}, \mathbf{y}(\mathbf{x})) = F(\mathbf{x}, S(\mathbf{x}), \eta_1(\mathbf{x}), \cdots, \eta_{n-1}(\mathbf{x})) = 0.$$

for all $\mathbf{x} \in V_1$.

Since γ, $\mathbf{N}_i, 1 \le i \le n-1$ are Lipschitz on $[0,\ell]$ and $\Gamma_m \to \gamma$ uniformly and $\mathbf{N}_{i,m} \to \mathbf{N}_i$ uniformly on $[0,\ell]$ for all $1 \le i \le n-1$, there exists an open interval $O \subset \mathbb{R}$ containing t_0, an open

subset $\Delta \subset U_1$ containing θ_0 and an integer M such that for all $t \in [0, \ell] \cap O$, $\theta \in \Delta$ and $m \geq M$,

$$\mathbf{x}(t, \theta) = (\gamma(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \theta) \in V_1 \quad \text{and,}$$

$$\mathbf{x}_m(t, \theta) = (\Gamma_m(t), \mathbf{N}_{1,m}(t), \cdots, \mathbf{N}_{n-1,m}(t), \theta) \in V_1.$$

Apparently $\mathbf{x}_m(t, \theta) \to \mathbf{x}(t, \theta)$ uniformly for all $t \in [0, \ell] \cap O$ and $\theta \in \Delta$. Since S is C^1 on $\mathbf{x} \in V_1$,

$$S(\mathbf{x}_m(t, \theta)) \to S(\mathbf{x}(t, \theta)) \text{ uniformly on } t \in [0, \ell] \cap O \text{ and } \theta \in \Delta. \tag{A.3}$$

Moreover, since S is C^1 on $\mathbf{x} \in V_1$ and $\mathbf{x}$ is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$, S is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$.

Now note that since $F\big(\mathbf{x}(t, \theta), \mathbf{y}(\mathbf{x}(t, \theta))\big) = 0$ and $F\big(\mathbf{x}_m(t, \theta), \mathbf{y}(\mathbf{x}_m(t, \theta))\big) = 0$, for each $s = s(\theta) \in s(\Delta) \subset \mathbb{S}^{n-2}$, we have $S_s^\gamma(t) = S(\mathbf{x}(t, \theta))$ and $S_s^{\Gamma_m}(t) = S(\mathbf{x}_m(t, \theta))$. Thus by (A.3),

$$S_s^{\Gamma_m}(t) \to S_s^\gamma(t) \text{ uniformly on } t \in [0, \ell] \cap O \text{ and } s \in s(\Delta),$$

and $S_s^{\Gamma_m}(t)$ is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$.

It remains to observe that since $[0, \ell]$ and $\mathbb{S}^{n-2}$ are both compact they can be covered by a finite union of neighborhoods on which (A.3) holds. The proof is complete. □

Define

$$\lambda_m(t) := \min\Big\{1, 1/(\sup_{|s|=1}\{(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))\})\Big\}$$

where $\tilde{\kappa}_{i,m}, 1 \leq i \leq n-1$ are those found in Step 1. A first observation is $0 < \lambda_m \leq 1$. Indeed, there must exist $s \in \mathbb{S}^{n-2}$ such that $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0$ so the supreme over all $s \in \mathbb{S}^{n-2}$ must be nonnegative. On the other hand, $S_s^{\Gamma_m}$ as well as all $\tilde{\kappa}_{i,m}$ are bounded so λ_m is bounded below by a positive number.

A second observation is that λ_m is continuous. Indeed, by Lemma A.2, $(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))$ is uniformly continuous on $(s, t) \in \mathbb{S}^{n-2} \times [0, \ell]$. Hence the supreme over $\mathbb{S}^{n-2}$ is attained and a simple argument gives $h(t) := \sup_{|s|=1}\{(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t))\}$ is continuous.

We then define a vector valued function $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ as

$$(\tilde{\tilde{\kappa}}_{1,m}(t), \cdots, \tilde{\tilde{\kappa}}_{n-1,m}(t)) := \lambda_m(t)(\tilde{\kappa}_{1,m}(t), \cdots, \tilde{\kappa}_{n-1,m}(t))$$

$\tilde{\tilde{\kappa}}_m$ is obviously continuous. It remains to show $\tilde{\tilde{\kappa}}_m$ satisfies (A.2). Indeed, for any $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t)) = \lambda_m (S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)).$$

If $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0$, then by the definition of λ_m,

$$\lambda_m(t)(S_s^{\Gamma_m}(t) + \frac{\rho}{2})\big(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)\big) \leq \min\big\{(S_s^{\Gamma_m}(t) + \frac{\rho}{2})\big(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)\big),\, 1\big\} \leq 1.$$

If $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) < 0$, then

$$\lambda_m(t)(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) < 0 \leq 1.$$

Thus (A.2) is satisfied.

Step 3. We want to show that $(\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m}) \to (\kappa_1, \cdots, \kappa_{n-1})$ a.e. Indeed, we know that $\tilde{\kappa}_m = (\tilde{\kappa}_{1,m}, \cdots, \tilde{\kappa}_{n-1,m}) \to (\kappa_1, \cdots, \kappa_{n-1})$ a.e.. Therefore, all we need to show is $\lambda_m \to 1 \quad$ a.e..

By possibly replacing λ_m by a subsequence, it suffices to prove $\lambda_m \to 1$ in measure. From the definition of λ_m, it is enough to show the Lebesgue measure of the set:

$$E_m = \{t \in [0, l], \exists s \in \mathbb{S}^{n-2}, (S_s^{\Gamma_m}(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) > 1\}$$

goes to zero. First by assumption, $L_s^{\gamma}(t) - S_s^{\gamma}(t) > \rho > 0$ and by Lemma 4.7, $L_s^{\gamma}(t)(\sum_{i=1}^{n-1} \kappa_i(t) s_i) \leq 1$, thus,

$$(S_s^{\gamma}(t) + \rho)(\sum_{i=1}^{n-1} s_i \kappa_i(t)) \leq 1, \tag{A.4}$$

for all $t \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if $\sum_{i=1}^{n-1} s_i \kappa_i(t) < 0$, (A.4) is obvious. If $\sum_{i=1}^{n-1} s_i \kappa_i(t)) \geq 0$,

$$(S_s^{\gamma}(t) + \rho)(\sum_{i=1}^{n-1} s_i \kappa_i(t)) \leq L_s^{\gamma}(t)(\sum_{i=1}^{n-1} \kappa_i(t) s_i) \leq 1$$

which again gives (A.4).

If $t \in E_m$, there is $s \in \mathbb{S}^{n-2}$ such that

$$\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) > \frac{1}{S_s^{\Gamma_m}(t) + \rho/2}.$$

Therefore all $t \in E_m$ and our choice of $s = s(t)$ as above, we have,

$$\begin{aligned} |\tilde{\kappa}_m(t) - \kappa(t)| &\geq \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) - (\sum_{i=1}^{n-1} s_i \kappa_i(t)) \\ &> \frac{\rho/2 + S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)}{(S_s^{\Gamma_m}(t) + \rho/2)(S_s^{\gamma}(t) + \rho)} \geq \frac{\rho/2 - |S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)|}{\rho^2/2}. \end{aligned}$$

By Lemma A.2,

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } s \in \mathbb{S}^{n-2} \text{ and } t \in [0, \ell],$$

then we can find m sufficiently large so that $|S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)| < \rho/4$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$. Since $\tilde{\kappa}_m \to \kappa$ a.e.,

$$\lim_{m\to\infty} |E_m| \leq \lim_{m\to\infty} |\{t : |\tilde{\kappa}_m(t) - \kappa(t)| \geq \frac{1}{2\rho}\}| = 0$$

which is what we wanted to show.

Step 4. Since $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ are continuous, for each m we can find κ_m smooth and $|\tilde{\tilde{\kappa}}_m - \kappa_m| \to 0$ uniformly on $t \in [0, \ell]$. Hence for m sufficiently large,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{4})(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1 \tag{A.5}$$

Step 5. We now define our desired curve γ_m. Given $\kappa_m = (\kappa_{1,m}, \cdots \kappa_{n-1,m})$ smooth as found in Step 4, and $\kappa_{i_j,m} \to \kappa_{i_j}$ found in step 1, we again solve the system of ODEs

$$\begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}$$

and denote by the orthogonal frame $(\gamma_m'(t), \mathbf{N}_{1,m}(t), \cdots \mathbf{N}_{n-1,m}(t))$ the unique solution with initial conditions $\gamma_m'(0) = \gamma'(0)$ and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. Moreover, by Lemma A.1, $(\gamma_m'(t), \mathbf{N}_{1,m}(t), \cdots \mathbf{N}_{n-1,m}(t)) \to (\gamma'(t), \mathbf{N}_1(t), \cdots \mathbf{N}_{n-1}(t))$ uniformly. Let

$$\gamma_m(t) = \gamma(0) + \int_0^t \gamma_m'(\tau) d\tau.$$

We claim γ_m satisfies for m sufficiently large,

$$(S_s^{\gamma_m}(t) + \frac{\rho}{8})(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1 \tag{A.6}$$

Indeed, by the same argument of Lemma A.2 using implicit function theorem, $S_s^{\gamma_m}$ also converges to S_s^{γ} uniformly. Together with Lemma A.2 we obtain $|S_s^{\gamma_m} - S_s^{\Gamma_m}| \to 0$ uniformly. Thus the claim follows from (A.5).

Step 6. Finally, we claim that orthogonal fronts satisfy $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0, \ell]$.

For γ_m and its moving frame $(\gamma_m', \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m})$ found in Step 5, let $\Phi_m : [0, \ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi_m(t, s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t).$$

Let $\Sigma^{\gamma_m} = \{(t, s) : \Phi_m(t, s) \in \overline{\Omega}\}$. By the same argument as Lemma 4.6, Φ_m maps Σ^{γ_m} onto $\overline{\Omega(\gamma_m)}$ where $\Omega(\gamma_m)$ is the subset of Ω covered by all orthogonal fronts $F_{\gamma_m}(t), t \in [0, \ell]$. By the same computation as in (4.20),

$$J_{\Phi_m}(t, s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t).$$

Let $d := \operatorname{diam}(\Omega)$, we claim that

$$1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \geq \min\{\rho/16d, 1/2\},$$

for all $(t, s) \in \Sigma^{\gamma_m}$. Indeed, if $\displaystyle\sum_{i=1}^{n-1} (s_i/|s|)\kappa_{i,m}(t) \geq 1/2d$, then by (A.6),

$$1 - |s| \Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \Big) \geq 1 - S_s^{\gamma_m}(t) \Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \Big) \geq \frac{\rho}{8} \Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \Big) \geq \frac{\rho}{8} \cdot \frac{1}{2d}.$$

If $\displaystyle\sum_{i=1}^{n-1} (s_i/|s|)\kappa_{i,m}(t) < 1/2d$, then

$$1 - |s| \Big(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \Big) > 1 - \frac{|s|}{2d} \geq \frac{1}{2}.$$

Hence, the claim follows. By Inverse function theorem due to Clarke [5], Φ admits a local Lipschitz inverse, actually a *global* Lipschitz inverse $\Phi_m^{-1} : \overline{\Omega(\gamma_m)} \to \Sigma^{\gamma_m}$ since the Jacobian is everywhere bounded below by a positive constant in Σ^{γ_m}. In particular, Φ_m is one-to-one on Σ^{γ_m}. This implies all orthogonal fronts $F_{\gamma_m}(t), t \in [0, \ell]$ meet outside $\overline{\Omega}$. The proof of Lemma 4.13 is complete. □

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